2009–05–11

Lecturer: Bengt E W Nilsson

Chapter 17: T-duality of closed strings

17.1: Lagrangian and Hamiltonian systems

that we are familiar with.

Duality: a transformation that relates two (very) different descriptions of the same physics. These transformations tend to be discrete. I don't know any systems where they are not.

What we will study here in string theory is T-duality: a theory on an $S^1(R)$-compactified manifold is T-dual to a theory on $S^1\left(\frac{\alpha'}{R}\right)$. This says in a sense that there is a smallest possible radius in physics.

A similar duality, called S-duality, exists for the coupling constant g_s: $g_s \leftrightarrow \frac{1}{g_s}$. Small and large couplings are related. This is related to AdS/CFT.

You can also mix S and T-duality to get U-dualities ($\sim$ M-theory!).

Dualities in some well-known systems:

1) Electromagnetism, with $j^\mu = 0$.

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0, \qquad & \nabla \times \boldsymbol{B} &= \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \\ \nabla \cdot \boldsymbol{B} &= 0, \qquad & \nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}. \end{aligned}$$

Duality: $(\boldsymbol{E}, \boldsymbol{B}) \rightarrow (-\boldsymbol{B}, \boldsymbol{E})$ gives back the same equations.

Comments: The equations of motion are duality invariant, or symmetric, but the Lagrangian is *not* invariant. The Hamiltonian is.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \sim \boldsymbol{E}^2 - \boldsymbol{B}^2$$

while $\mathcal{H} = \frac{1}{2}(E^2 + B^2) \rightarrow +\mathcal{H}$.

2) Harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{k x^2}{2} \rightarrow H \quad \text{if} \begin{cases} m \rightarrow \frac{1}{k}, \\ k \rightarrow \frac{1}{m}, \end{cases} \quad \begin{cases} p \rightarrow -x \\ x \rightarrow p \end{cases}$$

$\Rightarrow$ Commutation relations are not changed. This is a canonical transformation.

17.2: Winding states in closed strings

Consider a closed string moving in a target space with one direction $x^{25} \equiv x \in S^1(R)$. $x \sim x + 2\pi R$. In principle we have a torus.

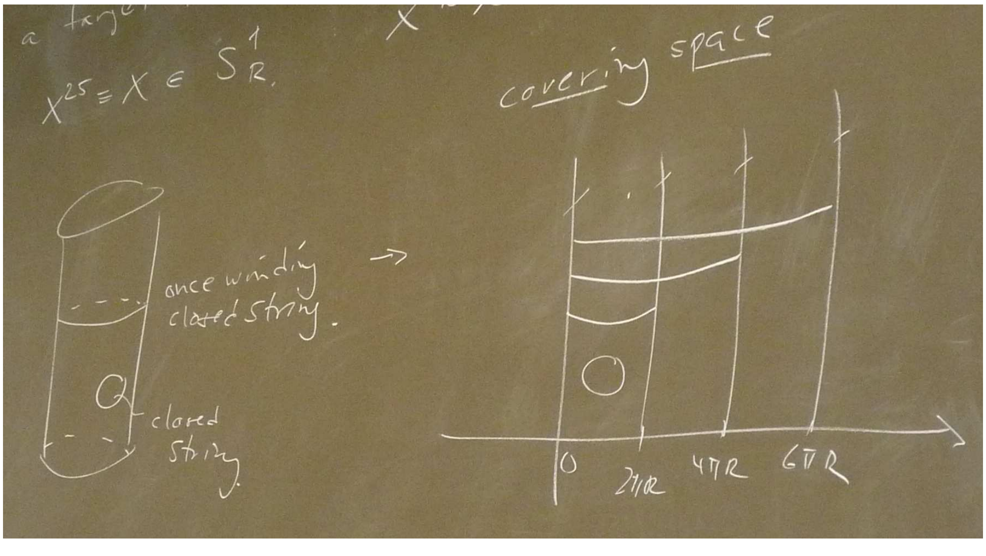

Figure 1.

These closed strings satisfy

$$x(\tau,\sigma+2\pi)=x(\tau,\sigma)+m\,2\pi R,\quad m\in\mathbb{Z}$$

m is called a winding number. Define another version of this number: winding $w = m\ R/\alpha'$ which has dimension $[w]=M$, as does momentum.

$$x(\tau,\sigma+2\pi)=x(\tau,\sigma)+2\pi\alpha'\cdot w$$

Expand x in modes.

$$\Box x=0 \quad\Rightarrow\quad x(\tau,\sigma)=x_L(\tau-\sigma)+x_R(\tau+\sigma)=x_L(u)+x_R(v)$$

Winding condition $\Rightarrow$

$$x_L(u+2\pi)-x_L(u)=x_R(v)-x_R(v-2\pi)+2\pi\alpha'\cdot w$$

So ∂_u and $\partial_\nu\Rightarrow x_L'$ and x_R' are 2π-periodic. Integrate

$$\Rightarrow\quad\begin{cases} x_L(u)=\frac{1}{2}x_{0,L}+\sqrt{\frac{\alpha'}{2}}\,\bar{\alpha}_0\,u+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n\mathrm{e}^{-\mathrm{i}n\,u}\\ x_R(v)=\frac{1}{2}x_{0,R}+\sqrt{\frac{\alpha'}{2}}\,\alpha_0\,v+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n\,\mathrm{e}^{-\mathrm{i}n\,u}\end{cases}$$

Inserting this into the winding relation

$$\Rightarrow\quad \bar{\alpha}_0-\alpha_0=\sqrt{2\alpha'}\,w$$

Also

$$p=\frac{1}{2\pi\alpha'}\int_0^{2\pi}(\dot{x}_L+\dot{x}_R)\mathrm{d}\sigma=\frac{1}{\sqrt{2\alpha'}}(\alpha_0+\bar{\alpha}_0)$$

$\Rightarrow\alpha_0$ and $\bar{\alpha}_0$ (or p and w) are independent! We find

$$\begin{cases} p=\frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}_0+\alpha_0)\\ w=\frac{1}{\sqrt{2\alpha'}}(\bar{\alpha}_0-\alpha_0)\end{cases}\qquad\begin{cases}\alpha_0=\sqrt{\frac{\alpha'}{2}}(p-w)\\ \bar{\alpha}_0=\sqrt{\frac{\alpha'}{2}}(p+w)\end{cases}$$

So we can define independent left and right momenta

$$\begin{cases} p_L=p+w\\ p_R=p-w\end{cases}$$

Write

$$\begin{cases} x_{0,L}\equiv x_0+q_0\\ x_{0,R}\equiv x_0-q_0\end{cases}$$

$$x(\tau,\sigma)=x_L(u)+x_R(v)=x_0+\alpha' p\,\tau+\alpha'\,w\,\sigma+\text{oscillators}(\alpha_n,\bar{\alpha}_n)$$

$[x_0,p]=\mathrm{i}$: canonical pair. Canonical coordiante to $w=?$

We need to understand the spectra of these (zero mode) operators:

• p: on a circle with radius r the translation operator is $\mathrm{e}^{-\mathrm{i}ap}$ (p is an operator here). Plug in $p = -\mathrm{i}\partial_x$ and $\mathrm{e}^{-\mathrm{i}ap} = \mathrm{e}^{-a\partial_x}$.

$$\mathrm{e}^{-\mathrm{i}ap}|x\rangle = \mathrm{e}^{-\mathrm{i}ap}\,\mathrm{e}^{-\mathrm{i}xp}|0\rangle = |x+a\rangle$$

But on a circle $x \in [0, 2\pi R]$.

$$\mathrm{e}^{-\mathrm{i}2\pi Rp} = 1.$$

The eigenvalues of the operator p are $n/R, n \in \mathbb{Z}$.

• w: Inser the mode expansion into the winding relation

$$\alpha'\,w(\sigma + 2\pi) = \alpha'\,w\,\sigma + m \cdot 2\pi R$$

$$\Rightarrow \quad \boxed{w = m\frac{R}{\alpha'}, \quad m \in \mathbb{Z}}$$

"Double strike" in Zwiebach: When x is compactified from $\mathbb{R} \rightarrow S^1(R)$:

1) we lose states when p becomes discrete!

2) we gain states, namely the winding states!

17.5: Constraints and mass formulae

Light-cone: $I = 2, \ldots, 25 = (i, (25))$.

$$\bar{L}_0^{\perp} = \frac{1}{2}\bar{\alpha}_0^I\bar{\alpha}_0^I + \bar{N}^{\perp} = \frac{\alpha'}{4}\,p^ip^i + \frac{1}{2}\bar{\alpha}_0\bar{\alpha}_0 + \bar{N}^{\perp}$$

$$L_0^{\perp} = \frac{1}{2}\alpha_0^I\alpha_0^I + N^{\perp} = \frac{\alpha'}{4}p^ip^i + \frac{1}{2}\alpha_0\alpha_0 + N^{\perp}$$

Level matching: $\bar{\alpha}_0^- = \alpha_0^-$ related to $L_0^{\perp}$ by the constriant

$$\Rightarrow \quad \bar{L}_0^{\perp} = L_0^{\perp}$$

$$\Rightarrow \quad \frac{1}{2}\bar{\alpha}_0\,\bar{\alpha}_0 + \bar{N}^{\perp} = \frac{1}{2}\alpha_0\alpha_0 + N^{\perp}$$

Use

$$\begin{cases} \alpha_0 = \sqrt{\frac{\alpha'}{2}}(p - w) \\ \bar{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}(p + w) \end{cases}$$

$$\Rightarrow \quad N^{\perp} - \bar{N}^{\perp} = \alpha'\,p \cdot w$$

or in terms of eigenvalues

$$N^{\perp} - \bar{N}^{\perp} = m \cdot n$$

Mass formula:

$$M^2 = p^2 + w^2 + \frac{2}{\alpha'}\big(N^\perp + \bar{N}^\perp - 2\big)$$

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\big(N^\perp + \bar{N}^\perp - 2\big)$$

17.6: State space

Consider the state $|p^\mu, m, n\rangle$

1) Lowest lying state is $|p^\mu, 0, 0\rangle$. Here $m = n = 0, N^\perp = \bar{N}^\perp = 0$.

$$M^2 = -\frac{4}{\alpha'} \quad \text{tachyon}$$

2) $N^\perp = \bar{N}^\perp = 1, m = n = 0$

$$\Rightarrow \quad M^2 = 0, \text{ massless!}$$

These states are (μ, ν: $0, 1, \ldots, 24$)

$$\alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |p, 0, 0\rangle$$

$g_{\mu\nu}, B_{\mu\nu}, \phi$ in $D = 25$ spacetime.

$$\alpha_{-1}^\mu \bar{\alpha}_{-1} |p, 0, 0\rangle$$

That's A_μ in $D = 25$ (Electromagnetism).

$$\alpha_{-} \bar{\alpha}_{-1}^\nu |p, 0, 0\rangle$$

$\tilde{A}_\mu$ in $D = 25$, ($\widetilde{\text{EM}}$).

$$\alpha_{-1} \bar{\alpha}_{-1} |p, 0, 0\rangle$$

scalar!

These states you would get via Kaluza–Klein compactification for $D = 25 \rightarrow D = 25$

$$D = 26 \rightarrow g_{\text{MN}} = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\mu & \phi \end{pmatrix}$$

$$B_{\text{MN}} = \begin{pmatrix} B_{\mu\nu} & \tilde{A}_\mu \\ -\tilde{A}_\mu & 0 \end{pmatrix}$$

3) Massless states from m and/or $n \neq 0$?

If just one of m and n is non-zero $\Rightarrow N^\perp = \bar{N}^\perp$. If $N^\perp = \bar{N}^\perp \geqslant 1$ no massless states. If $N^\perp = \bar{N}^\perp = 0$ massless state for particular value of R (a scalar).

So try m and n both non-zero. Set $n = \pm 1$, $m = \pm 1$

$$\Rightarrow N^\perp - \bar{N}^\perp = 1, \quad \Rightarrow N^\perp = 1, \bar{N}^\perp = 0 \text{ interesting!}$$

$$M^2 = \frac{1}{R^2} + \frac{R^2}{\alpha'^2} - \frac{2}{\alpha'} = \left(\frac{1}{R} - \frac{R}{\alpha'}\right)^2$$

$M^2 = 0$ for $R = R^* = \sqrt{\alpha'}$: a smallest possible radius.

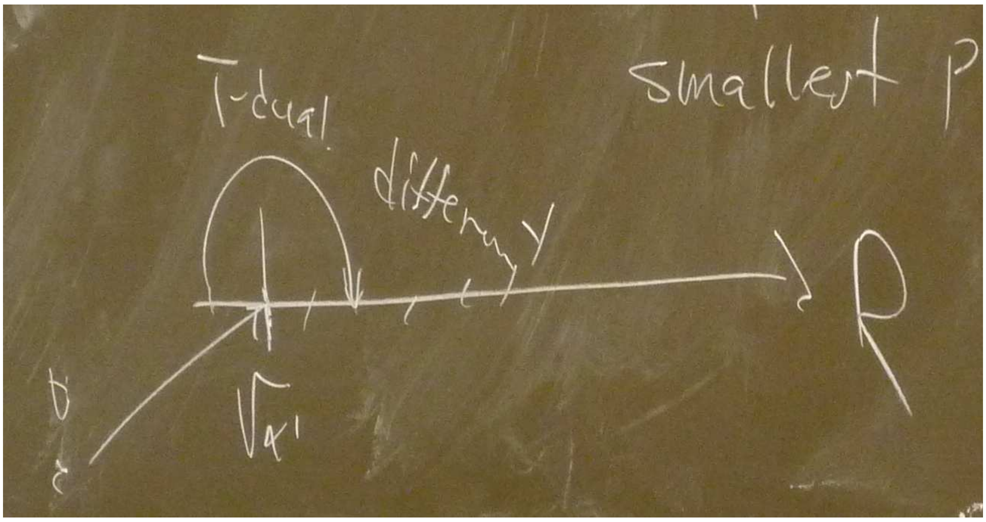

Figure 2.

What are the new massless states at R^*

$$\begin{cases} \bar{\alpha}^{\mu}_{-1}|p,\pm 1,\pm 1\rangle \\ \bar{\alpha}_{-1}|p,\pm 1,\pm 1\rangle \end{cases} \quad \begin{cases} \alpha^{\mu}_{-1}|p,\pm 1,\mp 1\rangle \\ \alpha_{-1}|p,\pm 1,\mp 1\rangle \end{cases}$$

i.e. four new vector states $A_\mu^{a=1,2,3,4}$ and four new scalar states.

Gauge symmetry: The two Kaluza–Klein vectors are U(1)'s (like electromagnetism). Adding the four other vectors we get $\mathrm{U}(1)\times\mathrm{U}(1)\to\mathrm{SU}(2)\times\mathrm{SU}(2)$. Symmetry enhancement.

17.7: Duality symmetry of the spectrum

T-duality: $R\leftrightarrow\tilde{R}=\alpha'/R$.

Fix point $R^*=\sqrt{\alpha'}$

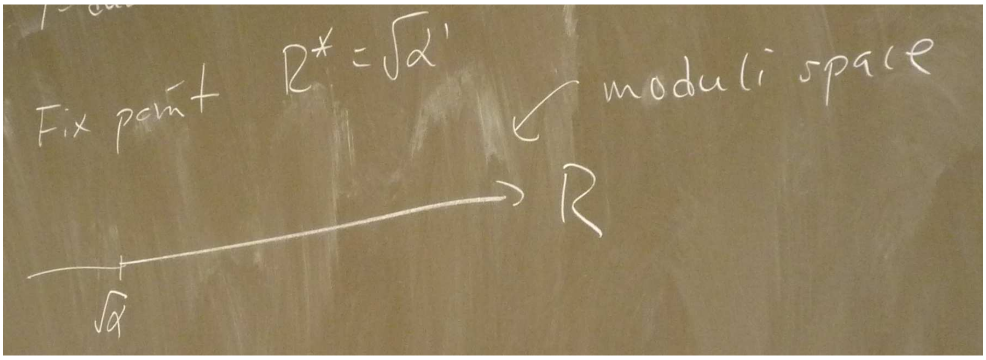

Figure 3. Moduli space

R is in fact the vacuum expectation value of a scalar field in string theory.

$$g_{MN}=\begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\mu & \phi \end{pmatrix}$$

Then

$$M^2(R,m,n)=M^2\Big(\tilde{R},n,m\Big)$$

$\Rightarrow$ T-dual physics is the same. (Can be proved also in the interacting theory.)

The "T" in T-dual stands for torii.