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Recap of the superstring

Recall the bosonic string

$$\begin{cases} \Box x^\mu = 0 & \Leftrightarrow \; \partial_+ \partial_- x^\mu = 0 \\ (\dot{x} \pm x')^2 = 0 & \Leftrightarrow \; (\partial_\pm x^\mu)^2 = 0 \end{cases}$$

The $\Box x^\mu = 0$ equation comes from

$$S \sim \int \mathrm{d}\tau \mathrm{d}\sigma \, \eta^{\alpha\beta} \, \partial_\alpha x^\mu \, \partial_\beta x^\nu \, \eta_{\mu\nu}$$

Where does the constraint come from? Either

1) we go back to $S \sim \int \; \mathrm{d}^2\sigma \sqrt{-g}$ (Nambu–Goto), or

2) Polyakov

$$S \sim \int \mathrm{d}\tau \, \mathrm{d}\sigma \, \sqrt{-h} \, h^{\alpha\beta} \partial_\alpha x^\mu \, \partial_\beta x^\nu \, \eta_{\mu\nu}$$

The constraint $\Leftrightarrow T_{\alpha\beta} = 0$ which is the $h_{\alpha\beta}$ field equation. (Traceless due to scale invariance).

The constraint is a consequence of coordinate invariance.

$$T_{\alpha\beta} = 0 \Rightarrow (\partial_\pm x^\mu)^2 = 0.$$

We repeat this strategy in the superstring case.

1) Introduce an action

$$S \sim \int \mathrm{d}\tau \mathrm{d}\sigma \big(\partial x \, \partial x + \bar{\psi} \rho^\alpha \partial_\alpha \psi \big)$$

This is supersymmetric (global symmetry). It produces

$$\begin{cases} \Box x^\mu = 0 \\ \rho^\alpha \partial_\alpha \psi^\mu = 0 \end{cases}$$

2) Then we need the constraints (to avoid negative-norm states in the Hilbert space). To be able to eliminate the effects of $\psi^{\mu=0}$ (negative-norm states), we need local supersymmetry. In other words: supergravity.

$$S = S[x^\mu, \psi^\mu, h_{\alpha\beta}, \chi_\alpha] \text{ with spin } 0, \, \frac{1}{2}, \, 2, \, \frac{3}{2}, \text{ respectively.}$$

$\delta h_{\alpha\beta}$ (Einstein's equations): $T_{\alpha\beta} = 0$ (in general $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$, but the left hand side is not present in two dimensions, for many reasons.) This gives us a constraint:

$$\text{“ } (\partial_\pm x^\mu)^2 \pm \frac{\mathrm{i}}{2}\, \psi_\pm^\mu \partial_\pm \psi_{\pm\mu} = 0 \text{ ”}$$

$\delta\chi_\alpha$ (Rarita-Schwinger equation) $J_\alpha = 0$ (supercurrent).

$$\Rightarrow \quad \psi_\pm^\mu \partial_\pm \chi_\mu = 0$$

Boundary conditions:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \text{ in Zwiebach.}$$

$$\psi_1^\mu \delta\psi_{\mu 1} - \psi_2^\mu \delta\psi_{\mu 2}\Big|_{\sigma=0}^{\sigma=\pi} = 0$$

$$\begin{array}{l} \text{at } \sigma = 0\text{:} \quad \psi_1 = \psi_2 \\ \text{at } \sigma = \pi\text{:} \quad \psi_1 = \pm\, \psi_2 \end{array}$$

$+$: We combine these into Ψ which is 2π periodic. This is called the Ramond sector.

$-$: Ψ is 2π anti-periodic. NS sector.

NS:

$$\Psi(\tau - \sigma) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r \mathrm{e}^{-\mathrm{i} r(\tau - \sigma)}$$

R:

$$\Psi(\tau - \sigma) = \sum_{m \in \mathbb{Z}} d_m \mathrm{e}^{-\mathrm{i} m(\tau - \sigma)}$$

Open superstring

$$M_{\mathrm{NS}}^2 = \frac{1}{\alpha'} \big(N_\alpha^\perp + N_b^\perp + a \big), \quad a = -\frac{1}{2}$$

$$M_{\mathrm{R}}^2 = \frac{1}{\alpha'} \big(N_\alpha^\perp + N_d^\perp + a \big), \quad a = 0$$

Here

$$N_\alpha^\perp = \sum_{n=1}^\infty \alpha_{-n}^I a_n^I, \quad \big[\alpha_m^I, \alpha_n^J\big] = m\, \delta_{m+n,0} \delta^{IJ}$$

$$N_b^\perp = \sum_{r=\frac{1}{2}}^\infty r\, b_{-r}^I b_r^I, \quad \big\{b_r^I, b_s^J\big\} = \delta_{r+s,0} \delta^{IJ}$$

$$N_d^\perp = \sum_{m=1}^\infty m\, d_{-m}^I d_m^I, \quad \{d_m^I, d_n^J\} = \delta_{m+n,0}\, \delta^{IJ}$$

Note: $\{d_0^I, d_0^J\} = \delta^{IJ}$ and that d_0^I does not appear in $N_d^\perp$. $\Rightarrow$ Combining the 8 d_0^I's into four creation and four annihilation operators we can construct $2^4 = 16$ degenerate vacuum states: i.e. the Ramond-vacuum is either a spinor $|s\rangle$ or a cospinor $|c\rangle$.

Closed superstring

Four sectors

(NS, NS)	(NS, R)	(R, NS)	(R, R)
bosnic states	fermionic	states	bispinor
	ex. $b^I_{-1/2}|s\rangle$		$|s\rangle \otimes |s\rangle$
			$=$ tensor $\Rightarrow$ bosonic states

RR-sector

IIA: $A^I, A^{IJK} \Rightarrow A_\mu, A_{\mu\nu\rho}$

IIB: $A, A^{IJ}, A^{IJKL(+)} \Rightarrow A, A_{\mu\nu}, A^{(+)}_{\mu\nu\rho\sigma}$. Compare $g_{\mu\nu}, B_{\mu\nu}, \phi$ from (NS, NS).

Chapter 15: D-branes and gauge fields

Consider the open superstring in $D = 10$ and configurations where they end on the same or different D-branes.

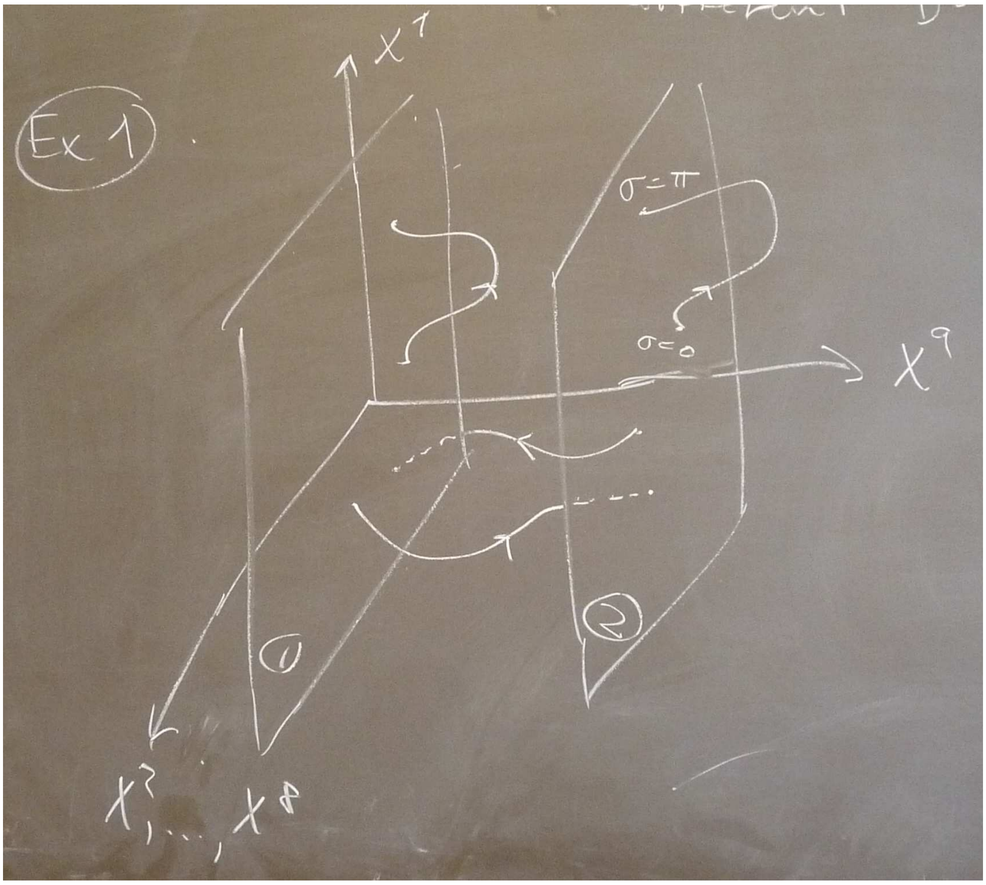

Figure 1.

Note: Two strings are identical (as particles in Quantum Mechanics), but D-branes can be “marked”.

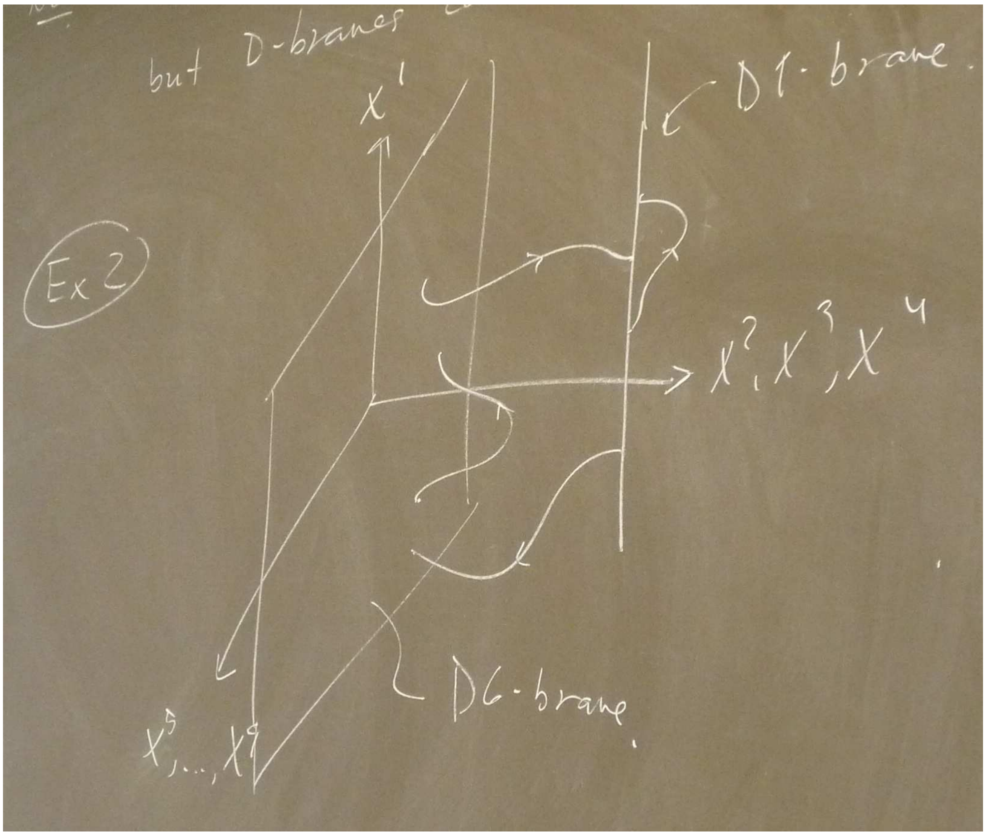

Figure 2.

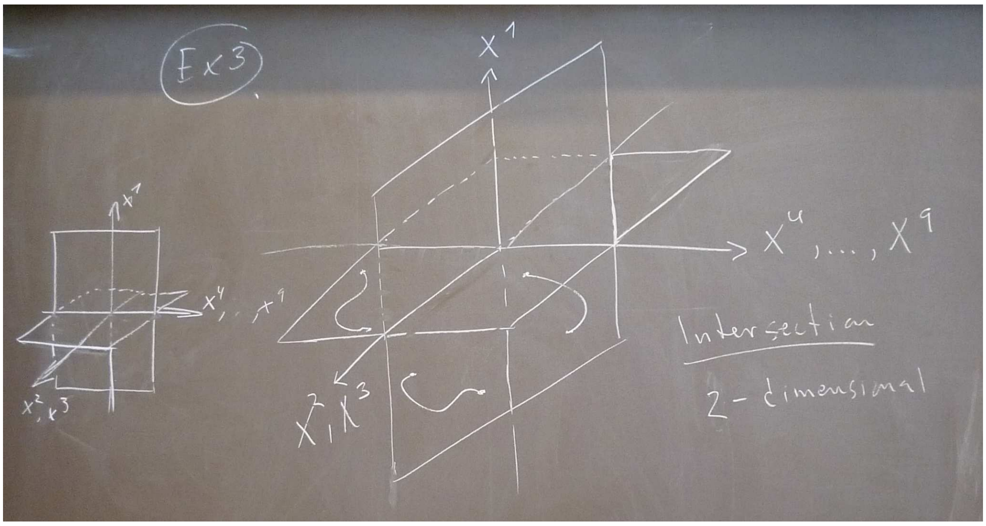

Figure 3. In the left figure the intersection is 6-dimensional, in the right one the intersection is 2-dimensional.

Example 1: Two cases: The two ends can end on

1) the same D-brane, or

2) on different, parallel, D-branes.

Let's consider Dp-branes.

$$\underbrace{x^0, x^1, x^2, \ldots, x^p}_{\substack{\text{in the brane} \\ x\text{'s N, N}}}, \underbrace{x^{p+1}, \ldots, x^d}_{\substack{\perp \text{ the D}p\text{-brane} \\ x\text{'s D, D}}}$$

$x^0, \ldots, x^p$ give expansion as usual. These x's we call x^+, x^-, x^i.

$x^{p+1}, \ldots, x^d$ give *new* expansions. These x's we call x^a, where $a = p+1, \ldots, d$.

For each x^a we have

$$x^a(\tau, \sigma)\big|_{\sigma=0} = x^a(\tau, \sigma)\big|_{\sigma=\pi} = \bar{x}^{\,a} = \text{fixed}$$

if the ends are on the same D-brane.

Expansion

$$\Box x^a = 0 \quad \Rightarrow \quad x^a(\tau, \sigma) = \frac{1}{2}(f^a(\tau+\sigma) + g^a(\tau-\sigma))$$

$\sigma = 0$ boundary condition

$$\Rightarrow \frac{1}{2}(f^a(\tau) + g^a(\tau)) = \bar{x}^{\,a}$$

$$\Rightarrow \quad g^a(u) = -f^a(u) + 2\bar{x}^{\,a}$$

$$\Rightarrow \quad x^a(\tau, \sigma) = \bar{x}^{\,a} + \frac{1}{2}(f^a(\tau+\sigma) - f^a(\tau-\sigma)): \quad \textit{note:} \text{ minus sign}$$

$\sigma = \pi$ boundary condition

$$\Rightarrow \quad \cancel{\bar{x}^a} + \frac{1}{2}(f^a(\tau+\pi) - f^a(\tau-\pi)) = \cancel{\bar{x}^a}$$

$$\Rightarrow f^a(\tau+\pi) = f^a(\tau-\pi)$$

$$\Rightarrow f^a \text{ is } 2\pi \text{ periodic.}$$

$$\Rightarrow x^a(\tau, \sigma) = \bar{x}^{\,a} + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a \mathrm{e}^{-in\tau} \sin n\sigma$$

Note: there is no term linear in $\tau \Rightarrow$ no momentum in this expansion. (OK: there is no motion in the x^a-directions.) Also, $\bar{x}^{\,a}$ is a number, not an operator.

Quantise: $\left[\alpha_m^a, \alpha_n^b\right] = m\,\delta^{ab}\,\delta_{m+n,0}$.

$$\Rightarrow \quad M^2 = -\,p^2 = 2p^+p^- - p^ip^i - p^a\!\!\!/p^a$$

$$2p^+p^- = \frac{1}{\alpha'}\big(L_0^\perp - 1\big)$$

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}^i\alpha_n^i + \sum_{n=1}^{\infty} \alpha_{-n}^a\alpha_{-n}^a - 1\right)$$

Comments: The fields (excitations) "live" on the D-brane.

The spectrum

- Has a tachyon.
- $M^2 = 0$ states of two types:

  $\alpha_{-1}^i|p\rangle \Rightarrow$ Gauge fields living on the brane.

  $\alpha_{-1}^a|p\rangle \Rightarrow$ Scalar on the brane (pointing $\perp$ the brane).
- $M^2 > 0$: infinitely many.

Let the string now end on two different branes:

$$\begin{cases} x^a(\tau,0) = \bar{x}_1^a \\ x^a(\tau,\pi) = \bar{x}_2^a \end{cases}$$

Repeating the story above:

$$x^a(\tau,\sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\sigma + \sqrt{2\alpha'}\sum_{n\neq 0} \frac{1}{n}\,\alpha_n^\alpha\,\mathrm{e}^{-int}\sin n\sigma$$

Still no momentum perpendicular to the branes.

$$M^2 = \frac{(\bar{x}_2^a - \bar{x}_1^a)^2}{(2\pi\alpha')^2} + \frac{1}{\alpha'}\Big(N_{(i)}^\perp + N_{(a)}^\perp - 1\Big)$$

So M^2 now depends on the distance between the branes. $M^2 = 0$ can occur in different ways.

Gauge fields? $\alpha_{-1}^i|p\rangle$

$$\Rightarrow \quad M^2 = \frac{(\bar{x}_2^a - \bar{x}_1^a)^2}{(2\pi\alpha')^2}$$

$\Rightarrow$ These vector fields $\alpha_{-1}^i|p\rangle$ are only massless (gauge fields) when $\bar{x}_2^a = \bar{x}_1^a$, i.e. when the branes are on top of each other (forming a "stack").

Higgs effect: the branes in a stack separate. The massless gauge field becomes massive, eating one of the scalars.

Non-abelian gauge fields. Consider again a stack of n D-branes.

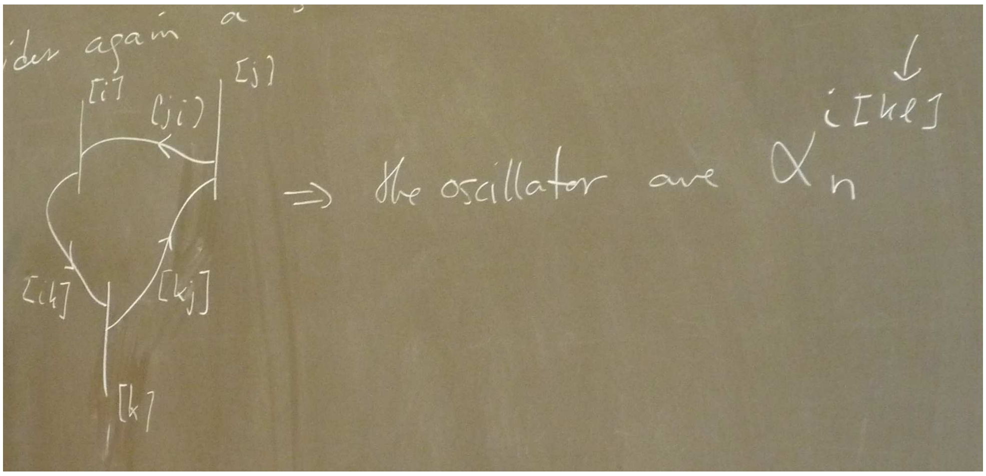

Figure 4. The oscillators are $\alpha_n^{i[kl]}$

We can now consider 3-point interaction:

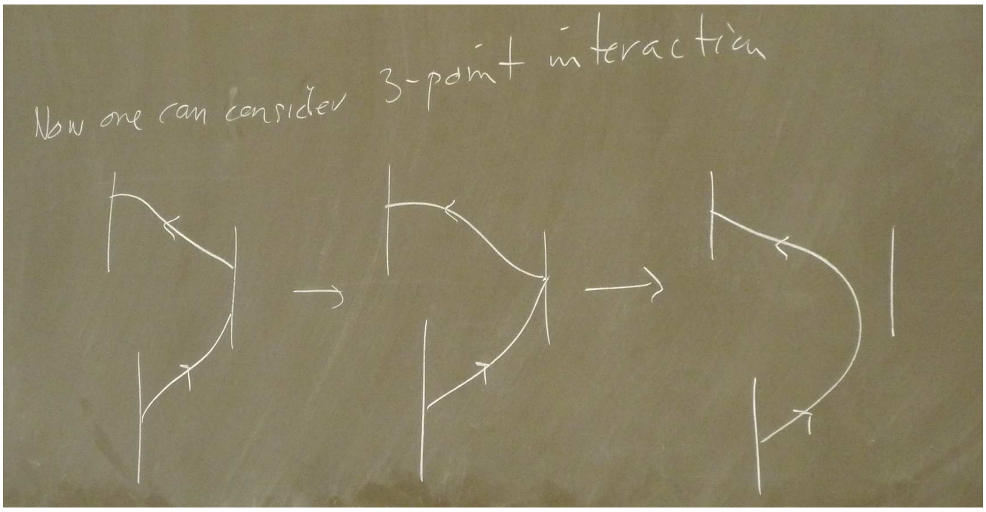

Figure 5.

i.e. $[ij]*[jk]=[ik]$. This is the 3-point interaction term in the Yang–Mill Lagrangian.

$$A_\mu \equiv A_\mu^A T^A$$

T^A: matrices. Generators of some Lie algebra.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$$

$$[T^A, T^B] = f^{AB}{}_C T^C$$

Gauge group?

2 D-branes: 2 U(1) + 2 string going between the branes = 4 branes $\Rightarrow$ U(2).

$$\mathrm{U}(2) \equiv \mathrm{SU}(2) \times \mathrm{U}(1)$$

N branes $\Rightarrow$ U(N) gauge group.

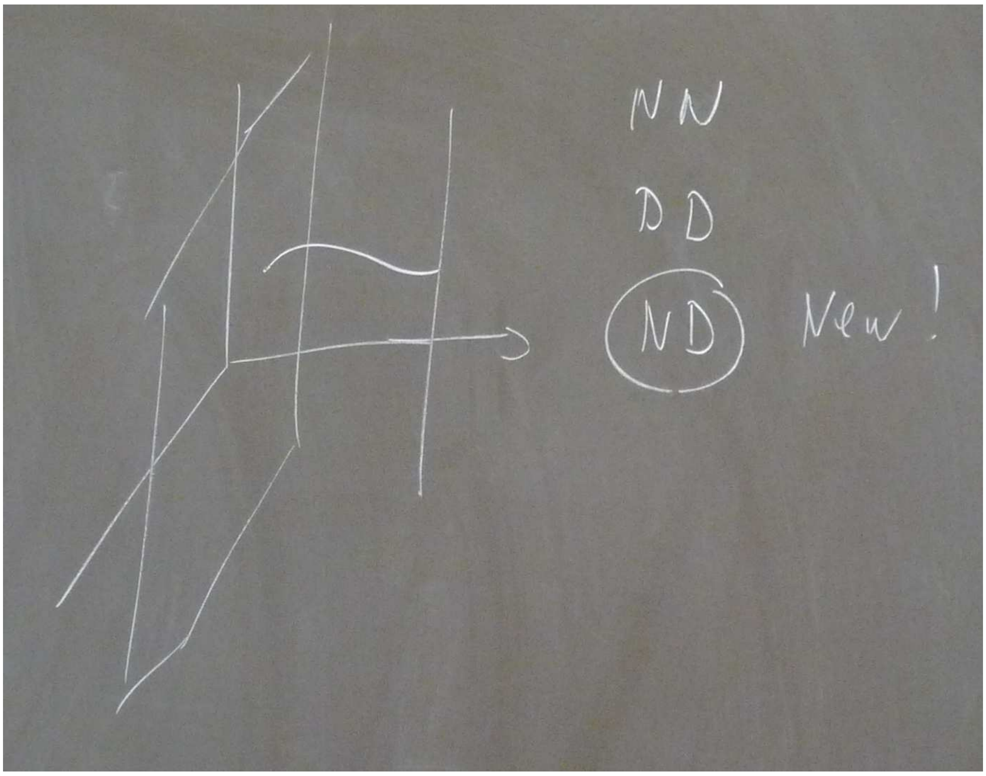

Figure 6.