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Chapter 13: The closed quantised bosonic string

1. Generalised τ, σ gauges: n_μ. For example $n_\mu = \frac{1}{\sqrt{2}}(1, 1, 0, ..., 0)$.

$$\begin{cases} n \cdot x = \alpha'(n \cdot p)\tau \\ (n \cdot p)\sigma = 2\pi \int_0^\sigma n \cdot P^\tau(\tau, \sigma')\mathrm{d}\sigma' \ \Rightarrow \sigma \in [0, 2\pi] \end{cases}$$

2. $(\dot{X} \pm X')^2 = 0$.

$$P^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu, \quad P^{\sigma\mu} = -\frac{1}{2\pi\alpha'}X'^\mu$$

$$\ddot{X}^\mu - X''^\mu = 0$$

3. Closed string periodicity $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$.

The Klein–Gordon equation implies

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(x - \sigma)$$

We define $u := \tau + \sigma$ and $v := \tau - \sigma$. You can rewrite $\ddot{X}^\mu - X''^\mu = 0$ as $\partial_u \partial_v X^\mu(u, v) = 0$.

So:

$$X_L(u + 2\pi) + X_R(v - 2\pi) = X_L(u) + X_R(u)$$

Then ∂_u, ∂_v: we find

$$\begin{cases} X_L'(u) \text{ is } 2\pi \text{ periodic} \\ X_R'(v) \text{ is } 2\pi \text{ periodic} \end{cases}$$

$$X_L'^\mu(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \bar{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}, \quad X_R'(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}$$

where α_n^μ and $\bar{\alpha}_n^\mu$ are independent Fourier coefficients (these were identified in the open case, due to boundary conditions). Integrate:

$$X_L^\mu(u) = \frac{1}{2}X_{0,L}^\mu + \sqrt{\frac{\alpha'}{2}}\,\bar{\alpha}_0^\mu u + \sum_{n \neq 0} \text{oscillators with a bar } (\bar{\alpha})$$

$$X_R^\mu(v) = \frac{1}{2}X_{0,R}^\mu + \sqrt{\frac{\alpha'}{2}}\,\alpha_0^\mu v + \sum_{n \neq 0} \text{oscillators}$$

But then the periodicity condition ($X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$) gives us $\alpha_0^\mu = \bar{\alpha}_0^\mu$. The momenta are the same in X_L and X_R. So, only one momentum $\Rightarrow$ only one centre of mass coordinate $\Rightarrow$ $X_{0,L}^\mu = X_{0,R}^\mu = X_0^\mu$.

So, closed string expansion in *Minkowski space*:

$$X^\mu(\tau,\sigma)=X_0^\mu+\sqrt{2\alpha'}\,\alpha_0\,\tau+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\cdot\left(\bar{\alpha}_n^{\,\mu}\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}+\alpha_n^\mu\mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}\right)$$

Note: If spacetime has a compactified direction then α_0 and $\bar{\alpha}_0$ will not be equal for that component of X^μ.

Note: If on an orbifold we can even a different type of expansion.

So, in the light-cone we have the following independent and canonical commutation relations:

$$\begin{aligned}
&\left[x_0^-,p^+\right]=-\,\mathrm{i}\\
&\left[x_0^I,p^J\right]=\mathrm{i}\,\delta^{IJ}\\
&\left[\alpha_m^I,\alpha_n^J\right]=m\,\delta^{IJ}\delta_{m+n,0}\\
&\left[\bar{\alpha}_m^I,\bar{\alpha}_n^J\right]=m\,\delta^{IJ}\delta_{m+n,0}
\end{aligned}$$

all others $=0$.

13.2: The closed string Virasoro operators

Constraint

$$\dot{X}^-\pm X'^-=\frac{1}{\alpha'\cdot 2p^+}\left(\dot{X}^I\pm X'^I\right)^2$$

Define

$$\left(\dot{X}^I+X'^I\right)^2=4\alpha'\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\bar{\alpha}_{n-p}^I\bar{\alpha}_p^I\right)\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}\equiv 4\alpha'\sum_{n\in\mathbb{Z}}\bar{L}_n^\perp\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}$$

The same for $\left(\dot{X}^I-X'^I\right)^2$ in terms of $L_n^\perp$. So this means that

$$\boxed{\bar{L}_0^\perp=L_0^\perp}\qquad\text{(Level matching condition)}$$

since $\alpha_0^-\sim L_0^\perp,\bar{\alpha}_0^-\sim\bar{L}_0^\perp$ and $\alpha_0^-=\bar{\alpha}_0^-$!

Now

$$\begin{cases}L_0^\perp=\frac{\alpha'}{4}p^Ip^I+N^\perp\\ \bar{L}_0^\perp=\frac{\alpha'}{4}\,p^Ip^I+\bar{N}^\perp\end{cases}$$

$$\Rightarrow\quad\boxed{N^\perp=\bar{N}^\perp}$$

Also

$$M^2=-\,p^2=2p^+p^--\,p^Ip^I$$

$$\Rightarrow\quad\boxed{M^2=\frac{2}{\alpha'}\left(N^\perp+\bar{N}^\perp-2\right)}$$

13.3 Mass spectrum

1) $|p^+, p^I\rangle$ no oscillators $\Rightarrow N^\perp = \bar{N}^\perp = 0$ and

$$M^2 = -\frac{4}{\alpha'} \quad \text{tachyonic!}$$

2) $\alpha_{-1}^I \bar{\alpha}_{-1}^J |p^+, p^I\rangle$: $N^\perp = \bar{N} = 1$ and $M^2 = 0$ massless states. Can also be created in field theory by $a_{p^+ p^I}^{\dagger IJ}|0\rangle \longrightarrow$ metric if IJ is symmetric, Kalb–Ramond field if IJ are anti-symmetric, Dilaton if $I = J$.

So the massless fields are

$g_{\mu\nu}$	metric	$\rightarrow$ Einstein
$B_{\mu\nu}$	Kalb–Ramond	$\rightarrow$ Kalb–Ramond field theory
ϕ	dilaton	$\rightarrow$ dil.

$+$ couplings (including the tachyon).

3) infinite set of massive fields.

13.4: String coupling and the dilaton

Recall: The metric has a background value.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ is a perturbation of the background $\eta_{\mu\nu}$. Also written:

$$g_{\mu\nu} = \langle g_{\mu\nu}\rangle + h_{\mu\nu}$$

$$S \sim \frac{1}{\kappa^2} \int \mathrm{d}^4x\, R$$

$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \longrightarrow$

$$\int \mathrm{d}^4x \big(h \Box h + \kappa h^2 \Box h + \cdots\big)$$

κ is a coupling constant.

This can also happen for a scalar: the dilaton:

$$\phi(x) = \langle\phi\rangle + \varphi(x)$$

In string theory we put

$$g_s = \mathrm{e}^{\langle\phi\rangle}$$

g_s is the string coupling constant.

Recall the covariant formalism

$$S \sim \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\gamma}\, \gamma^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$$

(Induced metric $g_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$).

Partition function (path integral):

$$Z \sim \int \mathrm{D}(\gamma, x)\, \mathrm{e}^{-S^g[\gamma,x] - \int \mathrm{d}\tau \mathrm{d}\sigma \phi(x) R_{(2)}(\tau,\sigma)\sqrt{\gamma}}$$

Inserting $\phi(x) = \langle\phi\rangle + \varphi(x)$: The $\langle\phi\rangle$ dependence becomes

$$\int \mathrm{D}(\gamma, x)\, \mathrm{e}^{-\langle\phi\rangle \underbrace{\int \mathrm{d}\tau \mathrm{d}\sigma R_{(2)}(\gamma)}_{\sim\chi}} = \mathrm{e}^{-\langle\phi\rangle\chi} \int \mathrm{D}(\gamma, x)\, \mathrm{e}^{\cdots}$$

χ: Euler number.

$$\mathrm{e}^{-\langle\phi\rangle\chi} = (g_s)^{-\chi}$$

$$\chi = 2 - 2g = b_0 - b_1 + b_2 = 2 \quad \text{(The } g \text{ is called genus.)}$$

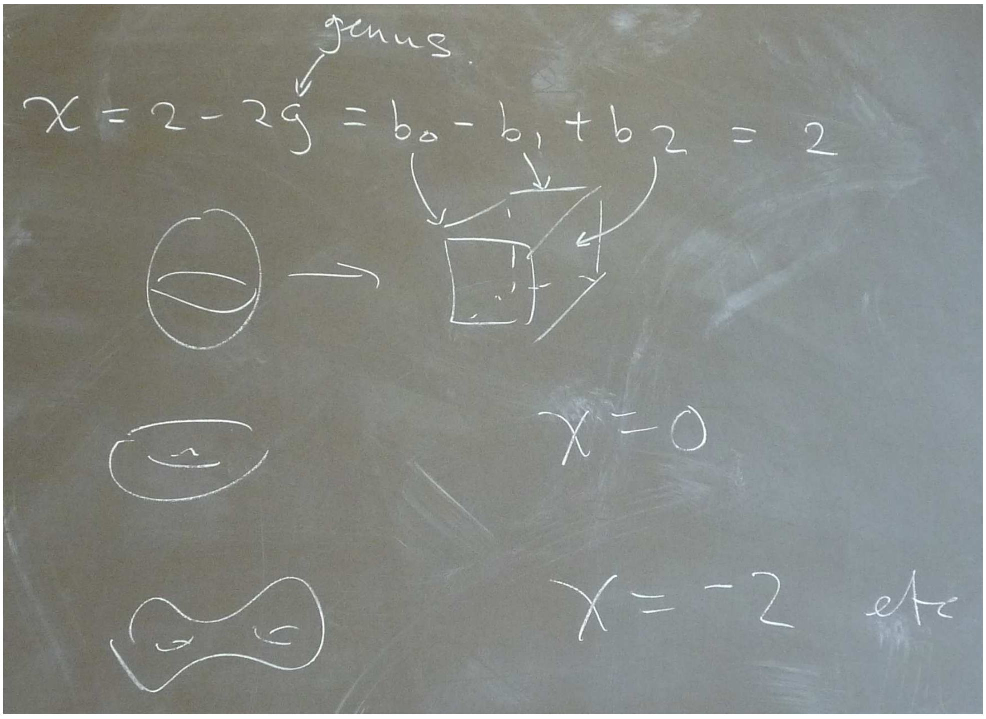

Figure 1. b_0: number of corners. b_1: number of edges. b_2: number of sides. The sphere is topologically a cube.

If we compute the $g = 0$ ($\chi = 2$) case (the S^2) we get

$$\mathcal{L} \sim \frac{1}{(g_s)^2 (\alpha')^4} R = \frac{1}{G_{\text{Newton}}} R$$

$$G_{\text{Newton}}^{D=10} = (g_s)^2 (\alpha')^4$$

13.5: Orbifolds

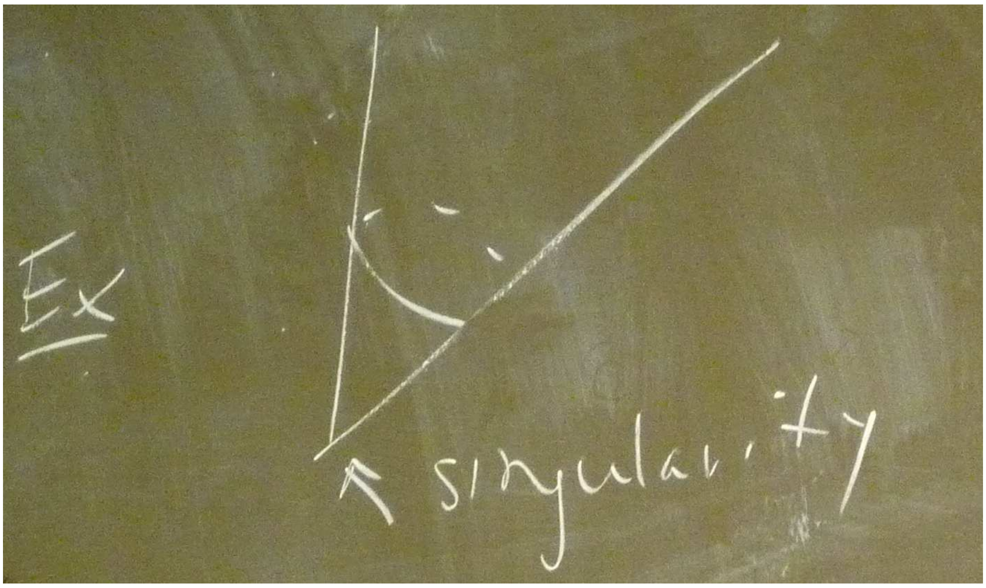

Figure 2.

A field theory cannot live on singular spaces, but string theory can. Consider a target space with a $\mathbb{Z}_2$ identification.

$$x^\mu\colon \quad \begin{array}{l} x^+ \to x^+ \\ x^- \to x^- \\ x^I \begin{cases} x^i \to x^i, & I = 2, \ldots, 24 \\ x \to -x, & I = 25 \end{cases} \end{array}$$

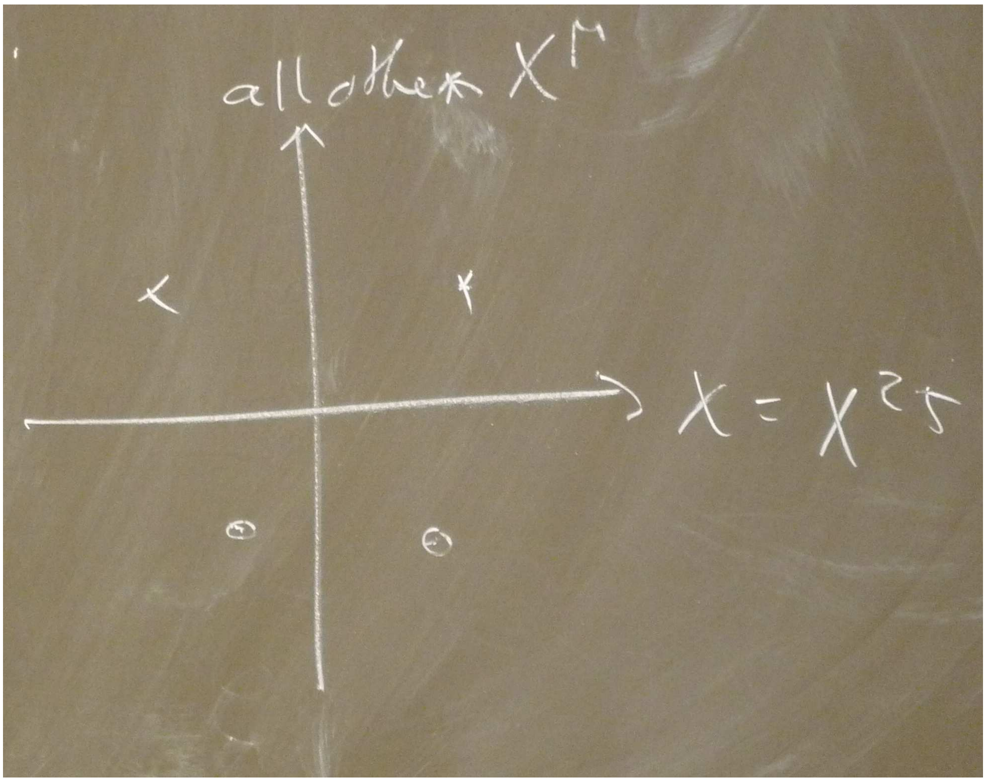

Figure 3.

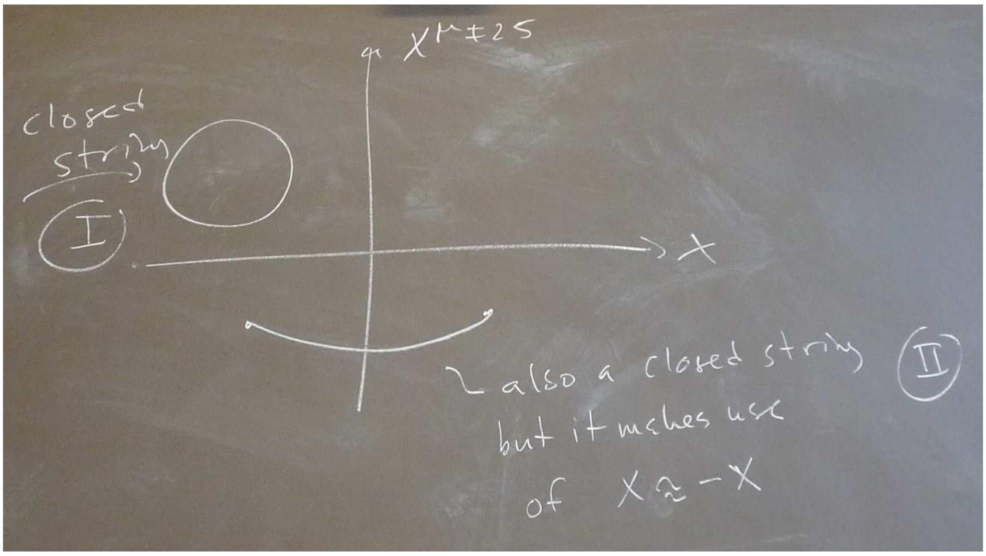

Figure 4.

The spectrum of the closed string on this orbifold has two sectors:

1) Untwisted. Spectrum = all states in the ordinary closed string invariant under $x \to -x$.

2) Twisted: A set of new states connected to the singularity (the fixpoint $x = 0$).

1) Mode expansions:

$$\begin{cases} x^{\mu\neq 25} & \text{as before} \\ x & \text{as before} \end{cases}$$

But we need an operator U

$$U X U^{-1} = -X$$

$$U X^{\mu\neq 25} U^{-1} = +X^{\mu=25}$$

$$U X_0 U^{-1} = -X_0, \quad U p U^{-1} = -p, \quad U \alpha_n U^{-1} = -\alpha_n, \quad U \bar{\alpha}_n U^{-1} = -\bar{\alpha}_n$$

$$\Rightarrow \quad U |p^+, p^i, p = 0\rangle = +|p^+, p^i, 0\rangle, \quad U |p^+, p^i, p\rangle = |p^+, p^i, -p\rangle$$

$$|p\rangle = \mathrm{e}^{\mathrm{i}\hat{x}_0 p} |0\rangle$$

$$\hat{p}|p\rangle = \hat{p}\,\mathrm{e}^{\mathrm{i}\hat{x}_0 p}|0\rangle = \left[\hat{p}, \mathrm{e}^{\mathrm{i}\hat{x}_0 p}\right]|0\rangle = [\hat{p}, ?]\mathrm{e}^{\mathrm{i}\hat{x}_0 p}|0\rangle = p|p\rangle$$

$$U\,|p\rangle = \underbrace{U\,\mathrm{e}^{\mathrm{i}\hat{x}_0 p} U^{-1}}_{= \mathrm{e}^{-\mathrm{i}\hat{x}_0 p}} \underbrace{U|0\rangle}_{=|0\rangle} = |-p\rangle$$

States invariant under $x \to -x$, i.e. U

1) $\quad |p^+, p^i, p\rangle + |p^+, p^i, -p\rangle$

2) $\alpha_{-1}^i \bar{\alpha}_{-1}^j \big(|p^+, p^i, p\rangle + |p^+, p^i, -p\rangle\big)$

$\alpha_{-1}^i \bar{\alpha}_{-1}\big(\,|p^+, p^i, p\rangle - |p^+, p^i, -p\rangle\,\big)$

13.6 Twisted sector

Mode expansion

$$\begin{cases} x^{\mu\neq 25} & \text{as before} \\ x & \text{new!} \end{cases}$$

$$x(\tau,\sigma+2\pi) = -x(\tau,\sigma)$$

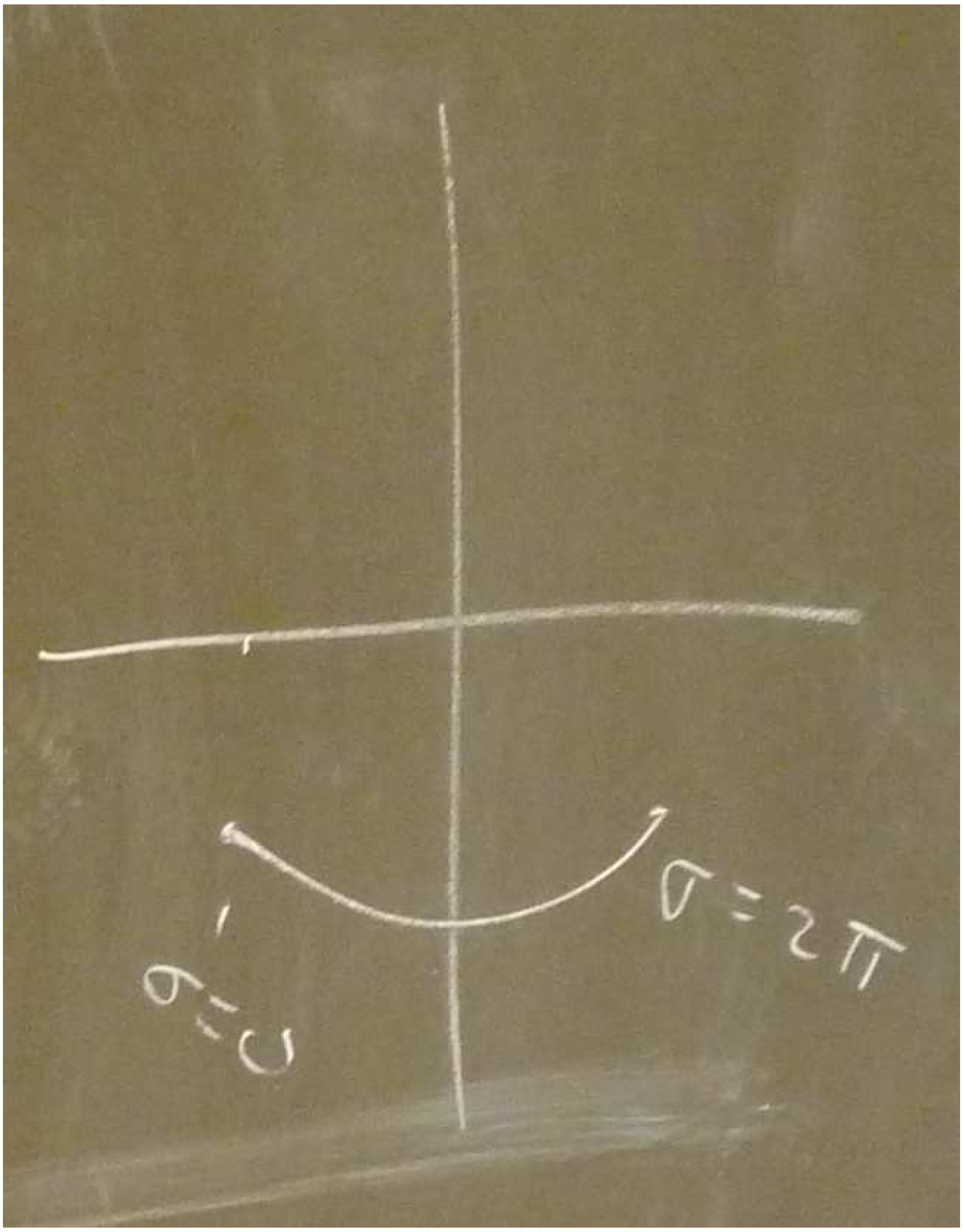

Figure 5.

$$X_L(u) = X_L + \mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_{\substack{r=\text{half}\\ \text{integer}}} \frac{1}{r}\,\bar{\alpha}_r\,\mathrm{e}^{-\mathrm{i}ru}, \qquad X_R(v) =$$

- No momenta (no mode with index = 0).
- No centre of mass coordinate either. ($X_L + X_R = 0$) The twisted sector is fixed to $x = 0$.