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Chapter 12: The relativistic quantum open string

- Derive the commutation relations for the modes in the expansion of x^I.
- Define the normal ordered transverse Virasoro generators.
- Compute the Virasoro algebra.
- Check the Lorentz algebra.

We will try to do it in two lectures.

Recall the result obtained using the general n_μ gauge from chapter 9. Klein–Gordon equation $\ddot{x}^\mu - (x'')^\mu = 0$, constraint $(\dot{x} \pm x')^2 = 0$, momenta $P^{\sigma\mu} = -\frac{1}{2\pi\alpha'}\, x'^{\,\mu}$, $P^{\tau\mu} = \frac{1}{2\pi\alpha'}\, \dot{x}^\mu$. Everything looks nice and covariant.

Now we go to the light-cone gauge. $n_\mu = (1, 1, 0, 0, \ldots, 0) \Rightarrow$

$$\begin{cases} x^+(\tau, \sigma) = 2\alpha'\, \tau \\ p^+\sigma = \pi \displaystyle\int_0^\sigma \mathrm{d}\sigma'\, P^{\tau +} \end{cases}$$

So $(\dot{x} \pm x')^2 = 0 \Rightarrow$

$$-2\left(\dot{x}^+ \pm x'^+\right)\left(\dot{x}^- \pm x'^-\right) + \left(\dot{x}^I \pm x'^I\right)^2 = 0$$

$x'^+ = 0$, $\dot{x}^+ = 2\alpha' p^+ \neq 0$.

$$\dot{x}^- \pm x'^- = \frac{1}{4\alpha'\, p^+}\left(\dot{x}^I \pm x'^I\right)^2$$

Add these two equations $\Rightarrow$

$$\dot{x}^+ = \frac{1}{4\pi p^+}\left(\left(\dot{x}^I\right)^2 + \left(x'^I\right)^2\right)$$

$$\Rightarrow \quad P^{\tau -} = \frac{\pi}{2p^+}\left(\left(p^{\tau I}\right)^2 + \frac{\left(x'^I\right)^2}{2\pi\alpha'}\right)$$

$\Rightarrow$ Independent canonical pairs are $\left(x^I(\sigma), P^{\tau I}(\sigma)\right)$, $\left(x_0^-, p^+\right)$.

So $\left[x_0^-, p^+\right] = -\mathrm{i}$.

$$\left[x^I(\sigma), P^{\tau J}(\sigma')\right] = \mathrm{i}\, \delta^{IJ}\, \delta(\sigma - \sigma')$$

All others $= 0$.

Hamiltonian.

$x^+ = 2\alpha'\, p^+ \tau$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial \tau}}_{\to H} = 2\alpha' p^+ \underbrace{\frac{\partial}{\partial x^+}}_{\to p^-}$$

$$\Rightarrow \quad H = 2\alpha' p^+ p^- = 2\alpha'\, p^+ \int_0^\pi \mathrm{d}\sigma\, P^{\tau -}(\sigma) =$$

$$= \pi\alpha' \int_0^\pi \mathrm{d}\sigma \left(P^{\tau I}(\tau, \sigma)\, P^{\tau I}(\tau, \sigma) + \frac{x'^I(\tau, \sigma) x'^I(\tau, \sigma)}{(2\pi\alpha')^2}\right)$$

This τ dependence is not really there. H is τ independent.

Note: from chapter 9:

$$L_n^{\perp} \equiv \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I a_p^I \quad \forall n$$

and

$$\sqrt{2\alpha'}\, \alpha_n^- = \frac{1}{p^+} L_n^{\perp}$$

In particular

$$L_0^{\perp} = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{-p}^I\, \alpha_p^I = \sqrt{2\alpha'}\, p^+ \alpha_0^- = 2\alpha' p^+ p^- = H$$

$\alpha_0^- = \sqrt{2\alpha'} p^-$.

So $H = L_0^{\perp}$. Note: In $L_0^{\perp}$ each term is ambiguous since α_p^I and α_{-p}^I don't commute! $\Rightarrow$ A proper definition later.

Equation of motion

$$\mathrm{i}\, \dot{\mathcal{O}} = [\mathcal{O}, H]$$

$$x^{\pm}(\tau, \sigma) \Rightarrow \mathrm{i}\, \dot{x}^I(\tau, \sigma) = \left[x^I(\tau, \sigma), H \right] =$$

$$= \pi\alpha' \int_0^{\pi} \mathrm{d}\sigma' \left[x^I(\tau, \sigma), P^{\tau J}(\tau, \sigma') P^{\tau J}(\tau, \sigma') + \frac{x'^J(\tau, \sigma') \cancel{x'^J}(\tau, \sigma')}{(2\cancel{\pi}\alpha')^2} \right]$$

The last term is zero since $[x^I(\tau, \sigma), x'^J(\tau, \sigma')] = \frac{\partial}{\partial \sigma'} \left[x^I(\tau, \sigma), x^J(\tau, \sigma') \right] = 0$.

$$\dot{x}^I(\tau, \sigma) = 2\pi\alpha'\, \dot{P}^{\tau I}(\tau, \sigma): \quad \text{OK.}$$

Do it for $P^{\tau I}$ also:

$$\dot{P}^{\tau I} = \frac{1}{2\pi\alpha'} \partial_\sigma^2(\tau, \sigma)$$

$$\dot{P}^{\tau I} = \frac{1}{2\pi\alpha'}\, \ddot{x}^I$$

This gives the Klein–Gordon equation. OK.

12.2: Commutation relations for oscillators

Recall

$$x^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau + \mathrm{i}\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n}\, \alpha_n^I \cos n\sigma\, \mathrm{e}^{-\mathrm{i}n\tau}$$

$$\Rightarrow \quad \begin{cases} \dot{x}^I = \sqrt{2\alpha'} \displaystyle\sum_{n \in \mathbb{Z}} \alpha_n^I \cos n\sigma\, \mathrm{e}^{-\mathrm{i}n\tau} \\ x'^I = -\mathrm{i}\sqrt{2\alpha'} \displaystyle\sum_{n \in \mathbb{Z}} \alpha_n^I \sin n\sigma\, \mathrm{e}^{-\mathrm{i}n\tau} \end{cases}$$

$$\Rightarrow \dot{x}^I \pm x'^I = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n \mathrm{e}^{-\mathrm{i}n(\tau \pm \sigma)}$$

valid for $\sigma \in [0, \pi]$.

However, the right-hand side has a natural periodicity 2π (not π), suggesting that

$$\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^I\,\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}=\begin{cases}\left(\dot{x}^I+x'^I\right)(\tau,\sigma) & \text{for}\,\sigma\in[0,\pi]\\ \left(\dot{x}^I-x'^I\right)(\tau,-\sigma) & \text{for}\,\sigma\in[-\pi,0]\end{cases}$$

Now we want to use this to get the canonical commutation relations for the oscillators. Start from

$$\left[x^I(\tau,\sigma),P^{\tau J}(\tau,\sigma')\right]=\mathrm{i}\,\delta^{IJ}\,\delta(\sigma-\sigma')$$

$$\Rightarrow[x^I(\tau,\sigma),\dot{x}^J(\tau,\sigma')]=2\pi\alpha'\,\mathrm{i}\,\delta^{IJ}\delta(\sigma-\sigma')$$

$\partial_\sigma\Rightarrow$

$$\left[x'^I(\tau,\sigma),\dot{x}^J(\tau,\sigma')\right]=2\pi\alpha'\,I\,\delta^{IJ}\,\partial_\sigma\delta(\sigma-\sigma')$$

We also have

$$[x',x]=0\quad\text{and}\quad\left[P^{\tau I},P^{\tau J}\right]\sim\left[\dot{x}^I,\dot{x}^J\right]=0$$

Then

$$\Big[\underset{(\tau,\sigma)}{\dot{x}\pm x'}\ ,\ \underset{(\tau,\sigma')}{\dot{x}\pm x'}\Big]=\pm 4\pi\mathrm{i}\,\alpha'\,\delta^{IJ}\partial_\sigma\delta(\sigma-\sigma')$$

and

$$[\dot{x}\pm x',\dot{x}\mp x']=0.$$

Insert these results for $\sigma,\sigma'\in[0,\pi]$

$$[(\dot{x}+x')(\tau,\sigma),(\dot{x}+x')(\tau,\sigma')]=2\alpha'\sum_{n,m}\mathrm{e}^{-\mathrm{i}n(\tau+\sigma)-\mathrm{i}n(\tau+\sigma')}\left[\alpha_n^I,\alpha_m^J\right]=$$

$$=4\pi\alpha'\,\mathrm{i}\,\delta^{IJ}\partial_\sigma\delta(\sigma-\sigma')$$

Now we can the previous extension to $[-\pi,\pi]$ since in fact this equation is valid for the whole range: Use $\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\,\mathrm{e}^{\mathrm{i}n\sigma}=\delta_{n,0}$ etc. Then

$$\left[\alpha_n^I,\alpha_m^J\right]=2\pi\mathrm{i}\,\delta^{IJ}\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}\,\mathrm{e}^{\mathrm{i}n\sigma}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}n\sigma'}\frac{\mathrm{d}\sigma'}{2\pi}\partial_\sigma\delta(\sigma-\sigma')=2\pi\,\delta^{IJ}\delta_{m+n,0}$$

$$\left[\alpha_n^I,\alpha_m^J\right]=2\pi\,\delta^{IJ}\,n\,\delta_{n+m,0}$$

Defining

$$a_n^I\equiv\frac{1}{\sqrt{n}}\alpha_n^I,\quad n>0$$

$$a_n^{\dagger I}\equiv\frac{1}{\sqrt{n}}\,\alpha_{-n}^I,\quad n>0$$

12.3 Strings as harmonic oscillators

Note that if we use the action

$$S=\int\,\mathrm{d}\tau\mathrm{d}\sigma\mathcal{L}=\frac{1}{4\pi\alpha'}\int\,\mathrm{d}\tau\int_0^\pi\,\mathrm{d}\sigma\Big(\dot{x}^I\dot{x}^I-x'^Ix'^I\Big)$$

$\eta = \text{diag}(-1,1); \eta^{\alpha\beta}\partial_\alpha x^I \partial_\beta x^I$.

$$\Rightarrow P^{\tau I} = \frac{1}{2\pi\alpha'}\dot{x}^{\pm}: \quad \text{OK}$$

$$\Rightarrow \quad H = \int_0^\pi \mathrm{d}\sigma\left(\pi\alpha'\left(P^{\tau I}\right)^2 + \frac{1}{k\pi\alpha'}\left(x'^I\right)^2 \ldots\right): \quad \text{Correct}$$

12.4: Transverse Virasoro operators

So far we have understood x^I in detail (there is no constraint here). What about $x^+(\tau,\sigma)$ and $x^-(\tau,\sigma)$?

$$x^+: \quad x^+(\tau,\sigma) = 2\alpha' p^+ \tau$$

$$x^-: \quad x^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\,\alpha_0^-\,\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\,\mathrm{e}^{-\mathrm{i}n\tau}\cos n\sigma$$

Using $(\dot{x} \pm x')^2 = 0 \Rightarrow$

$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\,\alpha_p^I \equiv \frac{1}{p^+}L_n^\perp$$

$L_n^\perp$ are the transverse Virasoro generators! Note that $L_0^\dagger$ has been defined.

$$L_0: \quad \sum_{p\in\mathbb{Z}}\alpha_{-p}\alpha_p = \sum_{p=1}^{\infty}(\alpha_{-p}\,\alpha_p + \alpha_p\,\alpha_{-p})$$

If we change order of these oscillators in $L_0^\perp$:

$$\sum_{n=1}^{\infty}\alpha_n^I\alpha_{-n}^I = \underbrace{\sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I}_{=N} + \sum_{n=1}^{\infty}\underbrace{(D-2)}_{I\text{-index}}\cdot n$$

$$\sum_{n=1}^{\infty}\underbrace{(D-2)}_{I\text{-index}}\cdot n = (D-2)\sum_{n=1}^{\infty} n = \text{strange constant.}$$

Is this strange constant important? **Yes!**

$$M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}L_0^\perp - p^Ip^I$$

so the constant will affect the mass.

- $L_0^\perp$ will enter the Lorentz generators and the constant will affect the Lorentz algebra.

The way to keep track of this constant

$$a \equiv \frac{1}{2}(D-2)\sum_{n=1}^{\infty} n$$

We will define $L_0^\perp$ by normal ordering.

$$L_0^\perp \equiv \;:\frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_n^I\alpha_n^I: \;=: \left(\frac{1}{2}\alpha_0^I\alpha_0^I + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\alpha_n^{\pm}\alpha_n^I}_{\substack{\text{normal}\\\text{ordered}}} + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\alpha_n^I\alpha_{-n}^I}_{\substack{\text{not normal}\\\text{ordered,}\\\text{flip}}}\right):$$

Use $:\alpha_n^I \alpha_{-n}^I: \equiv \alpha_{-n}^I \alpha_n^I$ to get

$$L_0^\perp = \frac{1}{2}\,\alpha_0^I \alpha_0^I + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I$$

This means that a will appear explicitly in all formulae:

$$2\alpha'\,p^- \equiv \frac{1}{p^+}\big(L_0^\perp + a\big)$$

and

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I + a\right)$$

This quantum spectrum is discrete and its lowest mass value is determined by a. Using now the Riemann $\zeta(s)$:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

This is convergent for $\mathrm{Re}(s) > 1$. The Riemann $\zeta(s)$ has an analytic continuation to all s. Then it gives

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

$$\Rightarrow a = -\frac{1}{24}(D-2)$$

Lowest mass squared is negative: tachyon! It means that the theory is unstable.

Properties of $L_n^\perp$

$$\big(L_n^\perp\big)^\dagger = L_{-n}^\perp \quad \forall n$$

They generate a Lie algebra.

1) $\mathrm{U}(1), \mathrm{SU}(2), \mathrm{SU}(3), \ldots, E_8$ are finite Lie algebras.

2) Virasoro algebra is infinite-dimensional.

Let's try to get the algebra.

1. $\big[L_m^\perp, \alpha_n^J\big] =$, first for $m \neq 0$ and $n \neq 0$.

$$\big[L_m^\perp, \alpha_n^J\big] = \frac{1}{2}\sum_{p\in\mathbb{Z}} \big[\alpha_{m-p}^I \alpha_p^I, \alpha_n^J\big] = \frac{1}{2}\sum_{p\in\mathbb{Z}} \big(\big[\alpha_{m-p}^I, \alpha_n^J\big]\alpha_p^I + \alpha_{m-p}^I\big[\alpha_p^I, \alpha_n^J\big]\big) =$$

$$= \frac{1}{2}\sum_{p\in\mathbb{Z}} \big((m-p)\delta^{IJ}\delta_{m-p+n,0}\,\alpha_p^I\big) =$$

$$= \frac{1}{2}\big(-n\,\alpha_{m+n}^J - n\,\alpha_{m+m}^J\big) = -n\,\alpha_{m+n}^J$$

$$\Rightarrow \big[L_m^\perp, \alpha_n^J\big] = -n\,\alpha_{m+n}^J$$

(conservation in mode number).

2. Check: also true for $n = 0$ and $m = 0$.

3. $\left[L_m^{\perp}, x_0^I \right] = -\,\mathrm{i}\sqrt{2\alpha'}\,\alpha_m^I$.

4. For $m + n \neq 0$:

$$\left[L_m^{\perp}, L_n^{\perp} \right] = \frac{1}{2} \sum_p \left[L_m^{\perp}, \alpha_{n-p}^I\,\alpha_p^I \right] = \ldots = \frac{1}{2}(m - n) \sum_p \alpha_{m+n-p} \alpha_p =$$

No ordering problem when $m + n \neq 0$.

$$= (m - n)\, L_{m+n}^{\dagger}$$

i.e. $[L_m^{\perp}, L_n^{\perp}] = (m - n) L_{m+n}^{\perp}$ when $m + n \neq 0$.

How do we deal with the $m + n = 0$ case? We can convince ourselves that the full algebra must take the form

$$\left[L_m^{\perp}, L_n^{\perp} \right] = (m - n) L_{m+n}^{\perp} + A^{\perp}(m)\, \delta_{m+n,0}$$

$A^{\perp}(m)$ $\delta_{m+n,0}$: no operator here: just a number. This is called conformal anomaly or central charge. You can check that this is OK with Jacobi identities.