2009–04–21

Lecturer: Bengt E W Nilsson

Chapter 11. The relativistic quantum point particle

The issue is *quantisation*.

$$\begin{array}{rcl} [\ ,\ ]_{\mathrm{PB}} & \to & \frac{1}{\mathrm{i}\hbar}[\ ,\ ] \\ q, p & \to & \hat{q}, \hat{p} \end{array}$$

Covariant quantisation (chapter 24). Light-cone quantisation (in the rest of the book). Covariant quantisation is much more powerful, once you get the hang of it.

Recall the action for the point particle:

$$S = -m\int_{\mathrm{init}}^{\mathrm{final}} \mathrm{d}s = -m\int_{\tau_{\mathrm{i}}}^{\tau_{\mathrm{f}}} \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,\mathrm{d}\tau$$

for some parametrisation $x^\mu(\tau)$. We write this as

$$S = \int L\,\mathrm{d}t, \quad L = -m\sqrt{-\dot{x}^2}.$$

Canonical momentum:

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\,\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$$

If you square this, you get $p^2 + m^2 = 0$. This is a constraint in phase space. (Phase space is described by q, p.)

An aside: In the Dirac formalism for (1st class) constrained systems one obtains *constraints* as a consequence of gauge invariance.

Equation of motion: $\dot{p}^\mu = 0$. Follows from Euler–Lagrange equations:

$$\partial_\tau\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

Note $H \equiv p_\mu \dot{x}^\mu - L = 0$ here! This is kind of problematic.

In our case we will have to design a new Hamiltonian such that the proper time evolution is obtained.

Light-cone gauge:

$$x^+(\tau) = \frac{p^+\tau}{m^2}$$

So this fixes the τ-reparametrisation invariance! How do we see this fact? Consider a small reparametrisation $\tau \to \tau' = \tau + \lambda(\tau)$, small λ.

$$\delta\tau \equiv \tau' - \tau = \lambda(\tau)$$

$$\delta x^\mu(\tau) \equiv x^\mu(\tau') - x^\mu(\tau) = \lambda(\tau)\,\partial_\tau x^\mu(\tau)$$

So to stay in the gauge we must demand that $\delta x^+(\tau) = 0 = \lambda(\tau)\partial_\tau x^+(\tau) = \lambda(\tau)\ p^+/m^2$. This requires that $\lambda(\tau) = 0$. The gauge invariance is eliminated.

Now:

$$p^+ \stackrel{\text{def}}{=\!=} \frac{m\dot{x}^+}{\sqrt{-\dot{x}^2}} \underset{\text{cone}}{\overset{\text{light-}}{=\!=}} \frac{p^+}{m\sqrt{-\dot{x}^2}} \Rightarrow m\sqrt{-\dot{x}^\mu} = 1$$

or

$$\dot{x}^2 = -\frac{1}{m^2}$$

(compare to $p^2 + m^2 = 0$). Then

$$p^\mu = \frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^2}} = m^2\dot{x}^\mu$$

Also $\dot{p}^\mu = 0 \Rightarrow \ddot{x}^\mu = 0$.

How many independent canonical pairs (q, p) do we have in the light-cone gauge?

First: *covariantly* we have all components x^μ and p^μ's. But these are not all independent in the light-cone gauge. $(x^I,\ p^I)$ are OK, they are independent. $(x^-,\ p^+)$ is also OK. $(x^+,\ p^-)$ is dependent:

$$x^+(\tau) = \frac{p^+\tau}{m^2}, \quad p^- = \frac{1}{2p^+}\left(p^I p^I + m\right)$$

We need to understand the τ-dependence in detail: should we solve $\ddot{x}^- = 0$, i.e.

$$x^-(\tau) = x_0^- + \frac{p^-\tau}{m^2}\ ?$$

And then which of x_0^- and $x^-(\tau)$ is the independent canonical coordinate? This is related to the question of the Hamiltonian:

$$\frac{\partial}{\partial x^+} \to p^- \quad \text{like} \quad \frac{\partial}{\partial \tau} \to H \text{ (Schröodinger equation)}$$

But since

$$x^+ = \frac{p^+\tau}{m^2} \quad \Rightarrow \quad \frac{\partial}{\partial x^+} = \frac{m^2}{p^+}\frac{\partial}{\partial \tau}$$

we have

$$\frac{\partial}{\partial \tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+}.$$

The operator associated with $\frac{\partial}{\partial \tau}$ is H.

$$H = \frac{p^+}{m^2}p^- = \frac{1}{2m^2}\left(p^I p^I + m^2\right)$$

Now we need to understand the τ dependence and check that this H does its job, i.e. give rise to

$$\begin{cases} x^-(\tau) = x_0^- + \frac{p^-\tau}{m^2} \\ x^I(\tau) = x_0^I + \frac{p^I\tau}{m^2} \\ x^+(t) = +\frac{p^+\tau}{m^2} \end{cases}$$

11.2: Heisenberg and Schrödinger pictures in Quantum Mechanics

In classical mechanics, we have $q(t), \dot{q}(t) \Rightarrow L(q, \dot{q})$ or we have $q(t), p(t) \Rightarrow H(q, p)$. L equations:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0.$$

H equations:

$$\frac{\partial H}{\partial q} = \dot{p}, \quad \frac{\partial H}{\partial p} = -\dot{q}.$$

Total t dependence of some function of phase space: $f(t; q(t), p(t))$:

$$\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial q}{\partial t}\frac{\partial f}{\partial q} + \frac{\partial p}{\partial t}\frac{\partial f}{\partial p}$$

Insert Hamilton's equations:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} \equiv \frac{\partial f}{\partial t} + [f, H]_{\mathrm{PB}}$$

Quantise:

$$[q, p]_{\mathrm{PB}} \to \frac{1}{\mathrm{i}\hbar}[\hat{q}, \hat{p}]$$

$[q, p]_{\mathrm{PB}} = 1 \to \frac{1}{\mathrm{i}\hbar}[\hat{q}(t), \hat{p}(t)] = 1$. Equal time commutation relations. This is the Heisenberg picture (as in classical physics). Operators $\mathcal{O}_H(t)$ are time dependent, and states $|\psi\rangle_H$ are time independents.

In the Schrödinger picture we have operators $\mathcal{O}_S$ and states $|\psi(t)\rangle_S$, but we have

$$|\psi(t)\rangle_S = \mathrm{e}^{-\mathrm{i}Ht}|\psi\rangle_H$$

$$\Rightarrow \quad \mathcal{O}_H(t) = \mathrm{e}^{\mathrm{i}Ht}\mathcal{O}_S\,\mathrm{e}^{-\mathrm{i}Ht}$$

11.3: Quantisation of the point particle

(x^I, p^I):

$$\left[\hat{x}^I, \hat{p}^J\right] = \mathrm{i}\,\delta^{IJ}$$

$(\hat{x}_0^-, \hat{p}^+)$:

$$\left[\hat{x}_0^-, \hat{p}^+\right] = \mathrm{i}\eta^{-+} = -\mathrm{i}$$

Here we assume that $\hat{x}_0^-$ is the correct variable. Is this true?

We have used the H to produce the following τ dependence:

$$\ddot{x}^\mu = 0 \quad \Rightarrow \quad \begin{cases} x^-(\tau) = x_0^- + \frac{p^-\tau}{m^2}, & \text{where } x_0^-, p^- \text{ are } \tau \text{ independent} \\ x^I(\tau) = x_0^I + \frac{p^I\tau}{m^2}, & \text{where } x_0^I, p^I \text{ are } \tau \text{ independent} \end{cases}$$

Use

$$\mathrm{i}\frac{\mathrm{d}\hat{f}}{\mathrm{d}t} = \mathrm{i}\frac{\partial \hat{f}}{\partial t} + \left[\hat{f}, \hat{H}\right]$$

First p^μ:

$$\mathrm{i}\dot{p}^I = \left[p^I, H\right] = 0 \quad \text{OK}$$

$$\mathrm{i}\dot{p}^+ = \left[p^+, H\right] = 0 \quad \text{OK}$$

$$\mathrm{i}\,\dot{p}^- = \left[p^-, H\right] = 0 \quad \text{OK}$$

Then x^μ:

$$\mathrm{i}\,\frac{\mathrm{d}x^I}{\mathrm{d}\tau} = \mathrm{i}\,\underbrace{\frac{\partial x^I}{\partial\tau}}_{=0} + \left[x^I, H\right] = \mathrm{i}\,\frac{p^I}{m^2}$$

τ dependency is implicit.

$$\Rightarrow \dot{x}^I = \frac{p^I}{m^2}: \quad \text{OK}$$

x^-: First

$$\mathrm{i}\,\frac{\mathrm{d}x_0^-}{\mathrm{d}\tau} = \mathrm{i}\,\underbrace{\frac{\partial x_0^-}{\partial\tau}}_{=0} + \left[x_0^-, H\right] = 0.$$

x_0^- is τ independent.

Then

$$\mathrm{i}\,\frac{\mathrm{d}x^-}{\mathrm{d}\tau} = \mathrm{i}\,\frac{\partial x^-}{\partial\tau} + \underbrace{\left[x^-, H\right]}_{=0}$$

i.e. the τ dependence in $x^-(\tau)$ is *explicit.*

Note: using x^- instead of x_0^- as independent does not work.

11.5: Light-cone momentum generators

Covariantly (i.e. all x^μ and p^μ are considered independent and satisfying $[x^\mu, p_\nu] = \mathrm{i}\delta^\mu{}_\nu$). Then since p^μ is the Noether charge for $\delta x^\mu = \varepsilon^\mu = \text{constant}$ (symmetry of $L = -m\sqrt{-\dot{x}^2}$).

We must have: (p^μ is the generator of the same transformation). i.e.

$$\delta x^\mu = [\mathrm{i}\,\varepsilon^\nu p_\nu, x^\mu] = \varepsilon^\mu$$

In the light-cone gauge all x^μ and p^μ are no longer independent $\Rightarrow$ check using ε^+ ($\varepsilon^- = \varepsilon^I = 0$):

$$\delta x^I = \mathrm{i}\left[-\varepsilon^+ p^- - \underbrace{\varepsilon^- p^+}_{=0} + \underbrace{\varepsilon^J p^J}_{=0}, x^I\right]$$

$p^- = \frac{1}{2p^+}\left(p^I p^I + m^2\right)$ does not commute with x^I. In fact

$$\delta x^I = -\frac{\varepsilon^+}{p^+}\,p^I$$

Strange.

Also

$$\delta x^+ = \left[-\mathrm{i}\varepsilon^+ p^-, x^+\right] = 0$$

Strange.

However, in the light-cone gauge $x^+ = p^+\tau/m^2$, we *want* $\delta x^+ = 0$, since we would like to keep the gauge condition intact.

So, what has happened? p^- has in fact two roles:

1) as a generator in x^+, i.e. as the charge giving $\delta x^+ = \varepsilon^+$.

2) as a generator in τ since is related to H.

So p^- produces the effect

$$\delta x^+ = \underbrace{\delta_\varepsilon x^+}_{\text{Noether}} + \underbrace{\delta_L x^+}_{\substack{\text{reparam.}\\ \delta\tau=\lambda(\tau)}} = \varepsilon^+ + \lambda\partial_\tau x^+ = \varepsilon^+ + \lambda\frac{p^+}{m^2} = 0$$

so that the light-cone gauge is not changed. Solve for λ:

$$\lambda = -\frac{m^2\varepsilon^+}{p^+}$$

This λ is a constant.

So this τ-reparametrisation is determined in terms of the Noether parameter ε^+: it is called a compensating reparametrisation!

11.6: Light-cone Lorentz generators

Covariant: $\delta^\mu = \varepsilon^\mu{}_\nu\, x^\nu$. The generator is

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu = (M^{\mu\nu})^\dagger$$

$$\Rightarrow \delta x^p = \left[-\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu} M_{\mu\nu}, x^\rho\right] = \varepsilon^{\rho\mu} x_\mu$$

Now we can view these in light-cone coordinates:

$$\left[M^{-I}, x^+\right] = -\,\mathrm{i}\,x^I$$

also

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\,\eta^{\mu\rho} M^{\nu\sigma} + 3\,\text{terms}$$

$$\left[M^{+-}, M^{+I}\right] = \mathrm{i}\,M^{+I}$$

$$\left[M^{-I}, M^{-J}\right] = 0$$

Light-cone gauge (x^μ, p^μ are no longer all independent)

Problems:

1) How is $M^{\mu\nu}$ defined in light-cone gauge.

2) What transformations do they generate?

3) Which algebra do $M^{\mu\nu}$ satisfy?

Answer 3: Lorentz algebra: has to be true in any gauge or formalism.

Answer 2: Lorentz transformations on x^μ and p^μ sometimes together with a compensating τ reparametrisation.

1?

Consider $M^{+-} = x^+(\tau)p^-(\tau) - x^-(\tau)p^+(\tau) =$

$$= \frac{p^+\tau}{m^2}\,p^- - \left(x_0^- + \frac{p^-\tau}{m^2}\right)p^+ = -\,x_0^-\,p^+ \quad \neq \quad \left(M^{+-}\right)^\dagger$$

So trick: Define

$$M^{+-} = -\frac{1}{2}\left(x_0^- p^+ + p^+ x_0^-\right)$$

Also M^{-I}:

$$M^{-I} \equiv x_0^- p^- - \frac{1}{2}\left(x^I p^- + p^- x^I\right)$$

And finally we have to check that the whole Lorentz algebra is OK using these M's. But there is a problem with

$$\left[M^{-I}, M^{-J}\right] = 0$$

This is home problem!

In string theory, demanding this commutator to vanish implies that the dimension is $D = 26$ and that the mass spectrums shifted so that the state corresponding to $g_{\mu\nu}$ is massless!