2009–03–23

Lecturer: Bengt E W Nilsson

Recall:

$$F = \frac{G\,M_1\,M_2}{r^2} \quad \Rightarrow \quad [G] = \frac{\mathrm{N\,m^2}}{\mathrm{kg^2}} \sim L^2 \text{ in natural units}$$

$$\Rightarrow l_{\mathrm{Pl}} = \sqrt{\frac{G\,\hbar}{c^3}}$$

This is in four dimensions. This scale l_{Pl} is the natural scale of quantum gravity.

§ 3.7: Gravitational potentials

$$\boldsymbol{F} = m\,\boldsymbol{g}, \quad \boldsymbol{g} = -\nabla V_g$$

(Valid in any dimension.)

$$\Rightarrow [V_g] = \frac{\text{energy}}{\text{mass}} = \text{dimension independent}$$

But then, in dimension $D = d+1$,

$$\underbrace{\nabla^2 V_g^{(D)}}_{D\text{-independent}} = 4\,\pi\,G^{(D)} \underbrace{\rho_m}_{\sim \frac{M}{L^{D-1}}}$$

(We will generally use a notation where D is the spacetime dimension, and d is the space dimension, so that $D = d+1$.)

$$\Rightarrow \left[G^{(D)}\right] = \left[G^{(4)}\right] L^{D-4}$$

where $G^{(4)} \equiv G =$ the ordinary Newton's constant.

§ 3.8: The Planck length in various dimensions

Definition:

$$l_{\mathrm{Pl}}^{(D)} := \left(\frac{\hbar\,G^{(D)}}{c^3}\right)^{\frac{1}{D-2}}$$

Look at the dimensions:

$$\left[\frac{\hbar\,G^{(D)}}{c^3}\right] = \left[\frac{\hbar\,G^{(4)}}{c^3}\right] L^{D-4} = L^2 \cdot L^{D-4} = L^{D-2}$$

So

$$\left(l_{\mathrm{Pl}}^{(D)}\right)^{D-2} = \frac{\hbar\,G^{(D)}}{c^3} = \left(l_{\mathrm{Pl}}^{(4)}\right)^2 \frac{G^{(D)}}{G^{(4)}}$$

$$\Rightarrow \quad G^{(D)} = \frac{\left(l_{\mathrm{Pl}}^{(D)}\right)^{D-2}}{(l_{\mathrm{Pl}})^2}\,G \quad \text{where}\, G^{(4)} \equiv G \text{ and } \; l_{\mathrm{Pl}} \equiv l_{\mathrm{Pl}}.$$

§ 3.9: The gravitational constant and compactification

Consider an example in $D=5$ where we have four non-compact spacetime dimensions and one dimension that is a circle (x^4), with a mass distribution

$$\rho^{(5)} = m\,\delta(x^1)\,\delta(x^2)\,\delta(x^3), \quad \text{with } [m] = \frac{M}{L} \text{ so that } \left[\rho^{(5)}\right] = \frac{M}{L^4}.$$

This is a mass spread evenly over the x^4 direction and is a point in x^1, x^2, x^3. Then

$$\nabla^2 V_g^{(5)} = 4\pi\, G^{(5)} \rho_m^{(5)}$$

where $V_g^{(5)} = V_g^{(5)}(x^1, x^2, x^3, \cancel{x^4}) \equiv V_g^{(4)}(x^1, x^2, x^3)$. Now

$$\rho_m^{(4)} \equiv \int_0^{2\pi R} \mathrm{d}x^4\, \rho_m^{(5)} = 2\pi\, R\, m\, \delta(x^1)\,\delta(x^2)\,\delta(x^3) = 2\pi\, R\, \rho_m^{(5)}$$

$$\Rightarrow \quad \rho_m^{(5)} = \frac{\rho_m^{(4)}}{2\pi R}.$$

$$\nabla_{(4)}^2\, V_g^{(4)}\,(x^1, x^2, x^3) = 4\pi\, G^{(5)}\, \frac{\rho^{(4)}}{2\pi R} \quad \Rightarrow \quad G^{(4)} \equiv \frac{G^{(5)}}{2\pi R}$$

$G^{(5)} = 2\pi\, R\, G^{(4)}$ and in D dimensions (repeat stepwise):

Defining $l_{\mathrm{c}} = 2\pi R$ (the compactification scale), then

$$G^{(D)} = (l_{\mathrm{c}})^{D-4}\, G = \{\text{volume of the } D-4 \text{ compact dimensions}\} \times G \equiv V_{\mathrm{c}}\, G$$

§ 3.10: Large extra dimensions

Two relations

- $l_{\mathrm{Pl}} \leftrightarrow$ gravitational constant, and
- $l_{\mathrm{c}} \leftrightarrow$ gravitational constant.

We can eliminate the gravitational constant:

$$(l_{\mathrm{c}})^{D-4} = \frac{\left(l_{\mathrm{Pl}}^{(D)}\right)^{D-2}}{(l_{\mathrm{Pl}})^2}$$

Note: l_{Pl} is the ordinary $D=4$ Planck constant, $l_{\mathrm{Pl}} \sim 10^{-35}\,\mathrm{m}$.

Example: Consider five dimensions: $D=5$. $l_{\mathrm{Pl}}^{(5)}$ — consider this as some kind of *fundamental scale*. So might be no bigger than today's experiments allow: $\leqslant 10^{-20}\,\mathrm{m}$.

Then

$$l_{\mathrm{c}} = \frac{(10^{-20})^3}{10^{-70}}\,\mathrm{m} = 10^{10}\,\mathrm{m}^2$$

That's crazy.

Try $D=6$:

$$l_{\mathrm{c}}^2 = \frac{10^{-80}}{10^{-70}}\,\mathrm{m}^2 = 10^{-10}\,\mathrm{m}^2, \quad l_{\mathrm{c}} = 10^{-5}\,\mathrm{m}$$

Meaning: Gravitationally, for $r \gg l_c$ we would have $F \sim r^{-2}$. For $r \ll l_c$, $F \sim r^{-4}$. $F \sim r^{-2}$ is known down to $55\,\mu$m.

What about using other forces? In string theory we can use *Brane World scenarios* to make a situation like this possible. In these cases the Standard Model with all the other forces, all forces except gravity, is confined to a $D=4$ world on a D-brane. This implies that these forces will not feel extra dimensions. This is a very strange concept, but it will become absolutely clear when we look at what the string is doing. Then only gravity can be used to discover extra dimensions. This entire idea is very tied to strings.

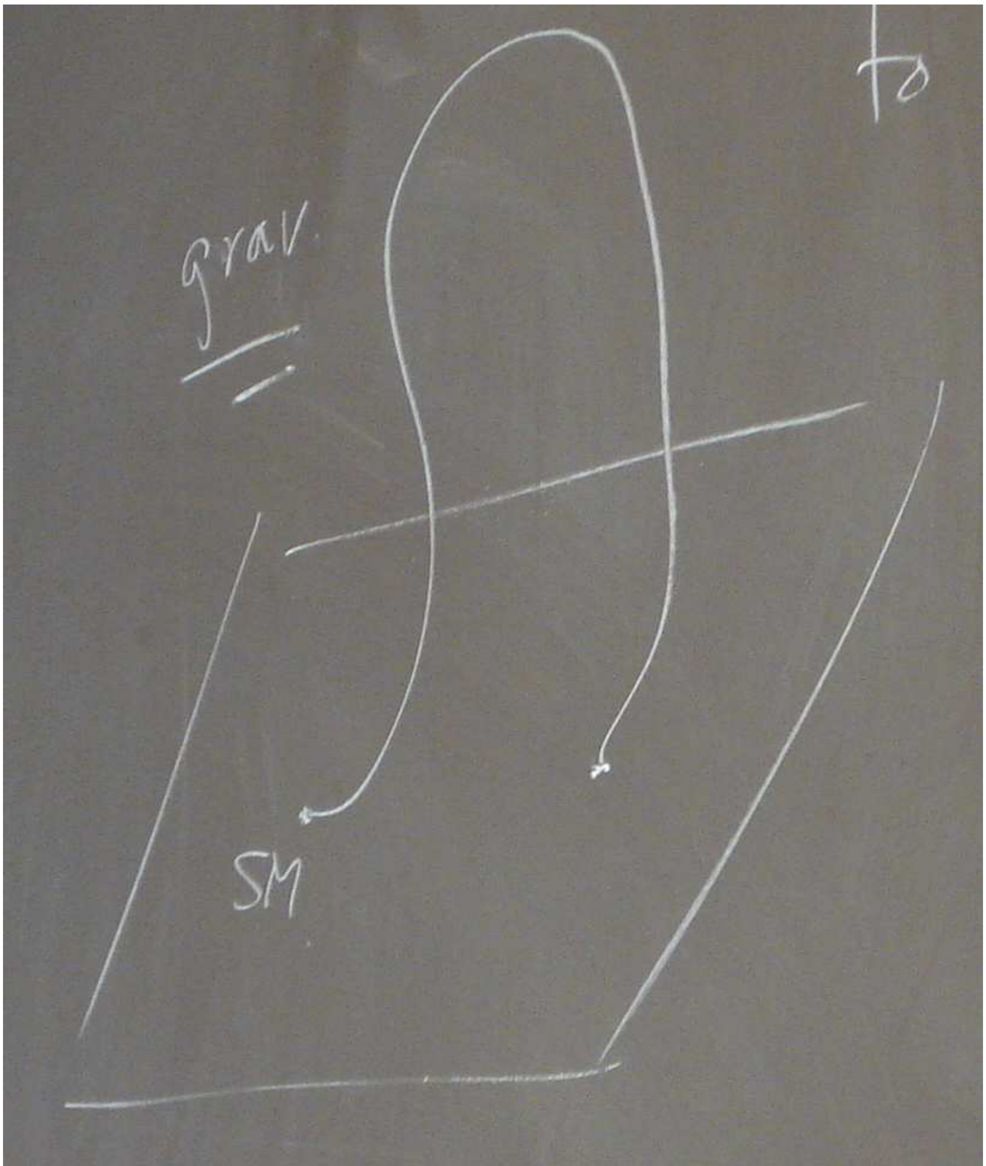

Figure 1. The Standard Model of particle physics is confined to a D-brane, i.e. the strings have both endpoints on the brane. Gravity gets to play with all the avaiable dimensions.

4. Non-relativistic strings

§ 4.1: Consider a string in two space dimensions, and it can oscillate only in the transverse direction (y-direction).

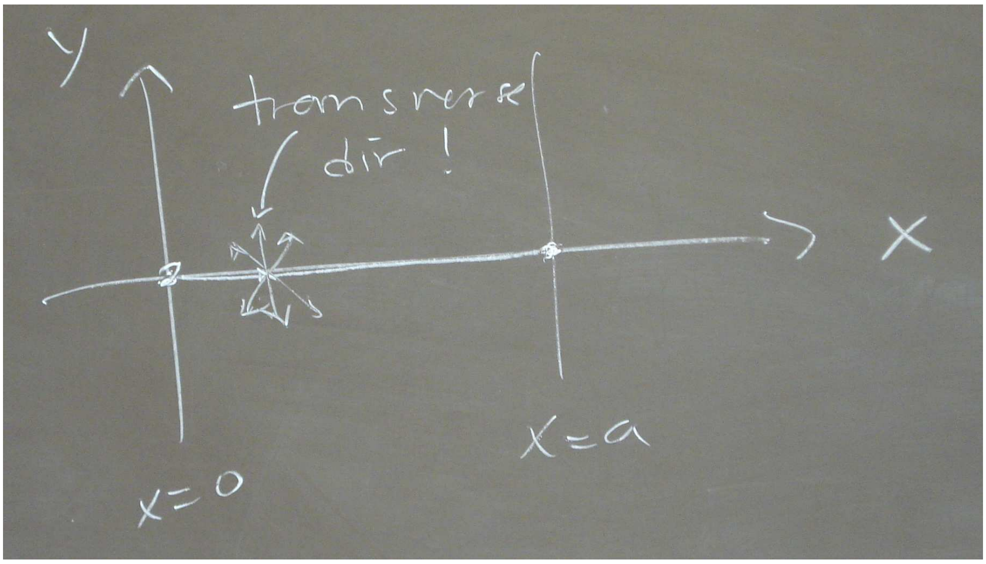

Figure 2.

Note: A *fundamental* string cannot oscillate longitudinally.

The mechanics of strings need T_0 (tension) and μ_0 (mass density). Total mass $M = \mu_0 \cdot a$.

Units:

$$[T_0] = [\text{Force}] = \frac{\text{N m}}{\text{m}} = \frac{\text{energy}}{\text{length}} \quad \Rightarrow \quad [T_0] = [\mu_0] \left[v^2\right]$$

- $T_0, \mu_0 \Rightarrow v$, the wave velocity. (T_0, μ_0 are independent in the non-relativistic case).
- Relativistic case: $T_0 = \mu_0 c^2 \Rightarrow T_0, \mu_0$ are dependent.

Equation of motion. (For a non-relativistic string.)

Figure 3.

Assume small oscillations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \ll 1$$

$\Rightarrow$ net horizontal force ≈ 0. The vertical force $=$

$$\mathrm{d}F_y = F_y(x+\mathrm{d}x) - F_y(x) = T_0\left(\left(\frac{\mathrm{d}y}{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}}\right)\bigg|_{x+\mathrm{d}x} - \left(\frac{\mathrm{d}y}{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}}\right)\bigg|_{x}\right) \approx T_0\,\frac{\partial^2 y}{\partial x^2}\bigg|_x \mathrm{d}x$$

Newton's equation for this piece of the string

$$\mathrm{d}F_y = \mathrm{d}m\,\ddot{y} = \mu_0\,\mathrm{d}x\,\ddot{y}$$

$$\Rightarrow \mu_0\ddot{y} = T_0\,\frac{\partial^2 y}{\partial x^2}$$

i.e.

$$\frac{\partial^2 y}{\mathrm{d}x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0, \quad \text{where } \frac{\mu_0}{T_0} = \frac{1}{v^2}, \text{with } v \text{ being the wave velocity.}$$

§ 4.2: Boundary conditions

To solve the wave equation we need two boundary conditions (in space), and two initial conditions (in time).

Boundary conditions.

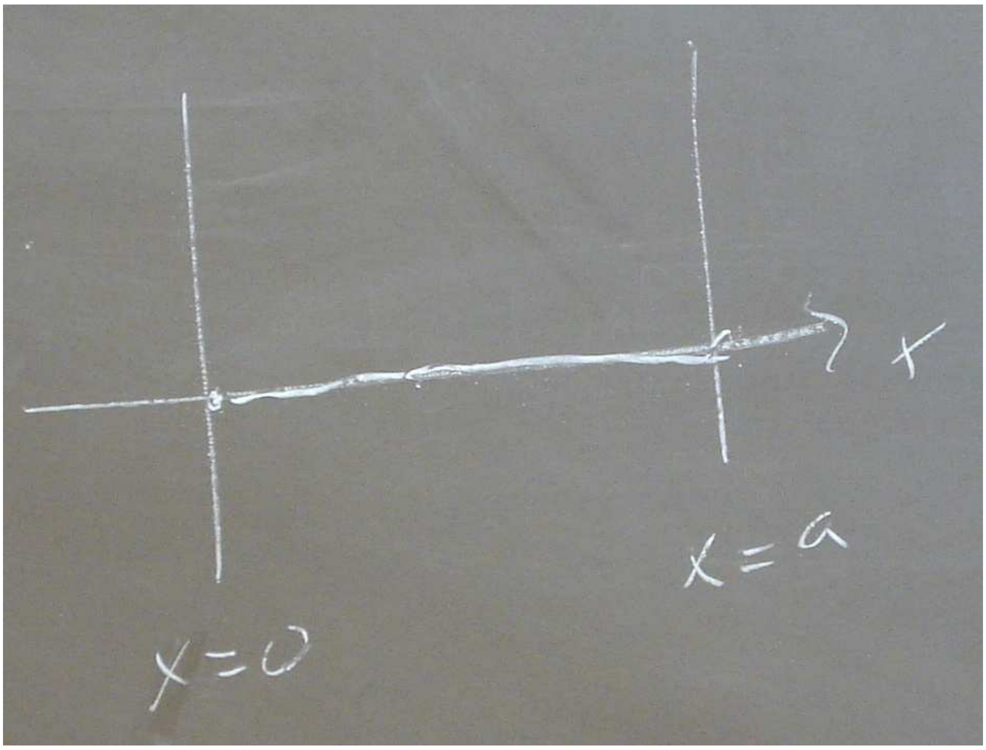

Figure 4. Dirichlet boundary conditions: fixed ends.

$$\begin{cases} y(t, x=0) = 0 \\ y(t, x=a) = 0 \end{cases}$$

Neumann boundary conditions: fixed slope.

$$\frac{\partial y}{\partial x}\bigg|_{x=0} = 0, \quad \frac{\partial y}{\partial x}\bigg|_{x=a} = 0$$

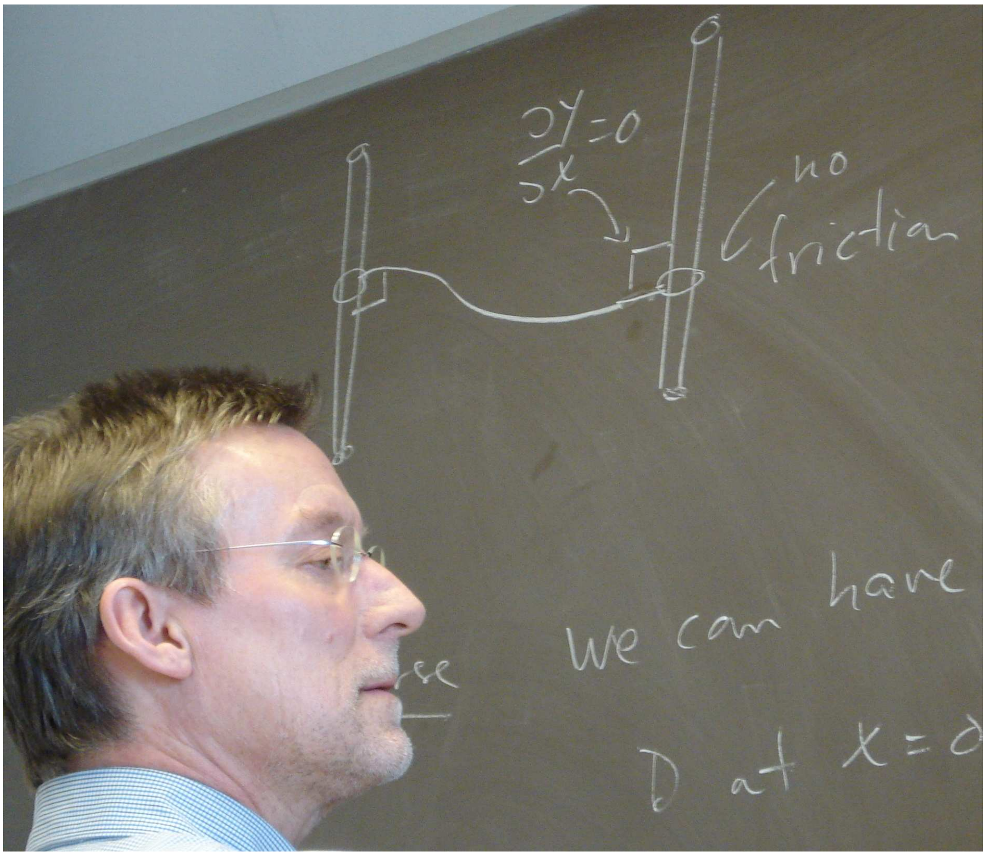

Figure 5. This setup gives Neumann boundary conditions, in the upper part of this image. We also get a glimpse of the lecturer himself (Bengt). He is writing about mixed boundary conditions as this picture is taken.

Of course, we can have mixed boundary conditions, e.g. Dirichlet at $x = 0$ and Neumann at $x = a$.

General solution:

Find $y(t, x)$ that solves

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial t^2} = 0.$$

In $D = 2$ we can write

$$x^{\pm} = x \pm v_0 t \quad \Rightarrow$$

$$\frac{\partial}{\partial x} = \frac{\partial x^+}{\partial x}\frac{\partial}{\partial x^+} + \frac{\partial x^-}{\partial x}\frac{\partial}{\partial x^-}$$

so

$$\begin{cases} \partial_x = \partial_+ + \partial_- \\ \frac{1}{v_0}\partial_t = \partial_+ - \partial_- \end{cases}$$

$$\partial_{\pm} = \frac{1}{2}\left(\partial_x \pm \frac{\partial_t}{v_0}\right)$$

$$\partial_+\partial_- = \frac{1}{4}\left(\partial_x^2 - \frac{1}{v_0^2}\partial_t^2\right)$$

So

$$y(t,x) = h_+(x - v_0\, t) + h_-(x + v_0\, t)$$

h_+ is called the right-moving term, and h_- is called the left-moving term.

§ 4.3: Example of oscillations

Consider $y(t,x) = y(x)\sin(\omega t + \varphi)$.

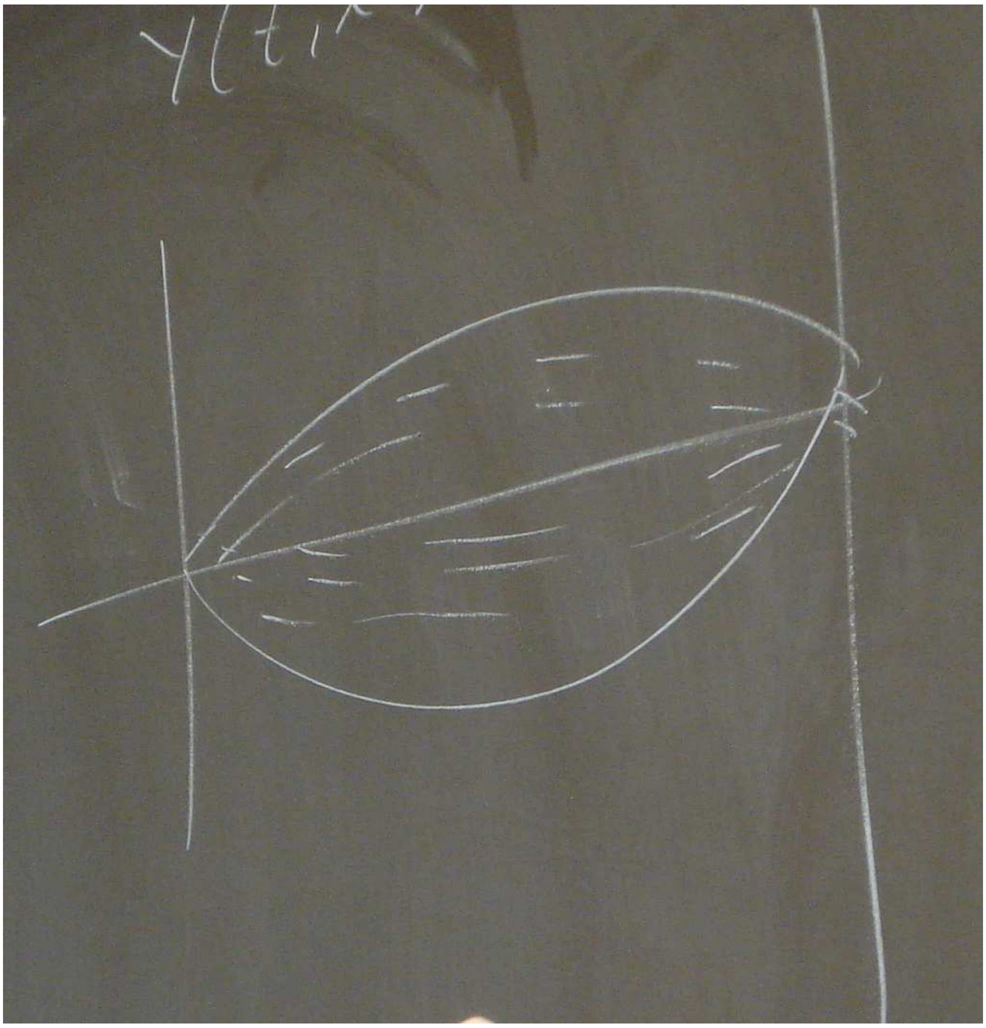

Figure 6.

Wave equation:

$$y'' + \frac{\omega^2}{v_0^2}\, y = 0$$

Solutions: $\cos(x), \sin(x)$.

Boundary conditions:

1) Dirichlet boundary conditions $\Rightarrow y_n(x) = A_n \sin\frac{n\pi\, x}{a}, n = 1, 2, \ldots$

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}} \cdot \frac{n\,\pi}{a}$$

2) Neumann boundary conditions (at both ends)

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = \left.\frac{\partial y}{\partial_x}\right|_{x=a} = 0$$

$$\Rightarrow \quad y_n(x) = B_n \cos\frac{n\,\pi\,x}{a}, \quad n = 0, 1, 2, \ldots$$

Note the zero mode $n = 0$.

$$\begin{cases} n \neq 0\text{:} \;\; y_n = B_n \cos\frac{n\pi x}{a} \\ n = 0\text{:} \;\; y_0 = a\,t + b \end{cases}$$

§ 4.5 Lagrangians

Why use Lagrangians?

- It gives dynamics from a variational principle.
- Very useful in quantum mechanics and quantum field theory.
- Lorentz and other symmetries are easy to implement. (L is a "scalar")
- Constrained systems (Dirac analysis).

DEFINITION:

$$L = T - V = E_{\text{kin}} - E_{\text{pot}}$$

Action $S = \int L\,\mathrm{d}t$.

Particle:

$$L = \frac{1}{2}\,m\,\dot{x}^2 - V(x)$$

Variations: (Hamilton's principle: dynamics is determined from variational principle.)

$$\delta S[x]$$

Functional, $x(t)$

$$\delta S[x] = S[x + \delta x] - S[x] = \int \mathrm{d}t\left(\frac{1}{2}\,m\left((\dot{x} + \delta\dot{x})^2 - \dot{x}^2\right) - (V(x + \delta x) - V(x))\right) =$$

$$= \int \mathrm{d}t\left(m\dot{x}\,\delta\dot{x} - \frac{\partial V}{\partial x}\,\delta x\right) = \underbrace{\int \mathrm{d}t\left(-m\,\ddot{x} - \frac{\partial V}{\partial x}\right)\delta x}_{\text{Bulk term}} + \underbrace{\int \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}\delta x)}_{\text{"Boundary term"}}$$

$$\begin{cases} \delta S = 0 \\ \delta S|_{t_{\mathrm{i}}} - \delta S|_{t_{\mathrm{f}}} = 0 \end{cases} \quad \Rightarrow \quad m\ddot{x} = -\frac{\partial V}{\partial x} \equiv F$$

This is Newton's equation.

§ 4.6: The non-relativistic string Lagrangian

$$L = T - V = \int_0^a \frac{1}{2}\,\mathrm{d}m\,\dot{y}^2 - \int_0^a T_0\,\mathrm{d}l$$

where $\mathrm{d}m = \mu_0\,\mathrm{d}x$ and $\mathrm{d}l = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} - \mathrm{d}x = \mathrm{d}x\sqrt{1 + \frac{\partial y^2}{\partial x^2}} - \mathrm{d}x \simeq \frac{1}{2}\,\frac{\partial y^2}{\partial x^2}\,\mathrm{d}x$. (Stretching from equilibrium position.

$$\Rightarrow L = \int_0^a \mathrm{d}x\left(\frac{1}{2}\,\mu_0\,\dot{y}^2 - \frac{1}{2}\,T_0\,y''\right) \quad \text{where} \begin{cases} y' \equiv \frac{\partial y}{\partial x} \\ \dot{y} \equiv \frac{\partial y}{\partial t} \end{cases}$$

$$\Rightarrow \quad S = \int \mathrm{d}t\,L = \int \mathrm{d}t \int \mathrm{d}x\,\mathcal{L}$$

where $\mathcal{L}$ is called the Lagrangian density. $\mathcal{L}$ describes a $D = 2$ field theory where y is the field and (t, x) are two-dimensional coordinates.

Variational principle

$$\mathrm{d}S = S[y + \delta y] - S[y] = \int \mathrm{d}t \int \mathrm{d}x(\mu_0 \dot{y}\,\delta\dot{y} - T_0\,y'\delta y') =$$

(integrate by parts in both x and t)

$$= \underbrace{\int \mathrm{d}t \int \mathrm{d}x(-\mu_0 \ddot{y}\,\delta y + T_0 y''\delta y)}_{\text{bulk term}} + \int \mathrm{d}t \int \mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}t}(\mu_0 \dot{y}\,\delta y) - \frac{\mathrm{d}}{\mathrm{d}x}(T_0 y'\delta y)\right) = 0$$

Bulk term:

$$\mu_0 \ddot{y} - T_0 y'' = 0 \quad \Rightarrow \quad y'' - \frac{\mu_0}{T_0}\,\ddot{y} = 0$$

Boundary terms: time direction: identically zero due to $\delta y|_{t_{\mathrm{i}}} = \delta y|_{t_{\mathrm{f}}} = 0$.

In the x-direction:

$$\int \mathrm{d}t\,y'\,\delta y|_{x=0} = \int \mathrm{d}t\,y'\delta y|_{x=a} = 0$$

Options at each end:

at $x = 0$:

1) $\delta y = 0 \Rightarrow$ Dirichlet (end is fixed)

2) $y' = \frac{\partial y}{\partial x} = 0 \Rightarrow$ Neumann

Note: Dirichlet can also be written

$$\frac{\partial y}{\partial t}\bigg| = 0$$

making it look like Neumann.

Total momentum conservation

$$p_y = \int_0^a \mathrm{d}x\,\mu_0\,\frac{\partial y}{\partial t}$$

$$\Rightarrow \dot{p}_y = \int_0^a \mathrm{d}x\,\mu_0\,\frac{\partial^2 y}{\partial t^2} = [\text{eq. of motion}] = \int_0^a \mathrm{d}x\,T_0\,\frac{\partial^2 y}{\partial x^2} = T_0\left[\frac{\partial y}{\partial x}\right]_{x=0}^{x=a} = \begin{cases} \text{Neumann} \Rightarrow = 0 \\ \text{Dirichlet} \Rightarrow \neq 0 \end{cases}$$

In the Dirichlet case the momentum at the ends is "taken up" by the walls to which the ends are attached.

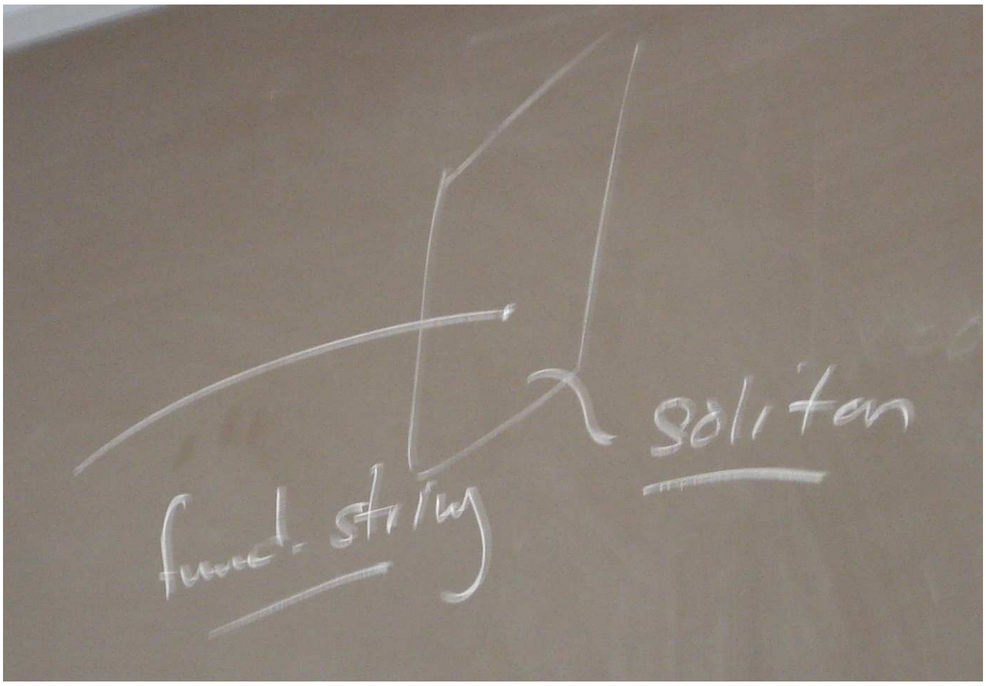

Figure 7. The fundamental string, attached to a soliton.