2008–10–02

$$\langle \boldsymbol{x}|\,[p_i, F(\boldsymbol{x})]\,|f\rangle =$$

$$= \langle \boldsymbol{x}| \quad (p_i F(\boldsymbol{x}) - F(\boldsymbol{x}) p_i) \quad |f\rangle =$$

$$= \int \mathrm{d}x' \,\langle \boldsymbol{x}|p_i|x'\rangle\langle x'|F(\boldsymbol{x})|f\rangle - F(\boldsymbol{x})\,\underbrace{(\langle \boldsymbol{x}|p_i|f\rangle)}_{= -\mathrm{i}\hbar\frac{\partial}{\partial x}\langle \boldsymbol{x}|f\rangle} =$$

$$= \int \mathrm{d}x' \left(-\mathrm{i}\,\hbar \frac{\partial}{\partial \boldsymbol{x}} \delta(\boldsymbol{x} - x') \right) F(\boldsymbol{x}') f(\boldsymbol{x}') =$$

$$= -\mathrm{i}\hbar \frac{\partial}{\partial \boldsymbol{x}} (F(\boldsymbol{x}) f(\boldsymbol{x})) - F(\boldsymbol{x}) \left(-\mathrm{i}\hbar \frac{\partial}{\partial \boldsymbol{x}} f(\boldsymbol{x}) \right) =$$

$$= -\mathrm{i}\hbar \left(\frac{\partial}{\partial \boldsymbol{x}} F(\boldsymbol{x}) \right) f(\boldsymbol{x})$$

6.1 N identical spin $\frac{1}{2}$ particles in a one-dimensional simple harmonic oscillator potential.

a) What is the ground state energy? What is the Fermi energy?

1 particle: $\left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$

$$p(E) = \frac{2}{1 + \mathrm{e}^{(E - E_f)/kT}}$$

$$N = \sum_i p_i = \sum_{E_i} \frac{2}{1 + \exp\left(\frac{E_i - E_f}{kT}\right)}; \quad E_i = \left(n_i + \frac{1}{2}\right)\hbar\,\omega$$

$$E_0 = 2\left\{\frac{1}{2}\hbar\,\omega + \frac{3}{2}\hbar\,\omega + \cdots + \left(\left[\frac{N}{2}\right] - \frac{1}{2}\right)\hbar\,\omega\right\} + \left\{\frac{N}{2} + \frac{1}{2}\right\}\hbar\omega\,\delta, \quad \delta = \begin{cases} 0 & \text{if } N \text{ even} \\ 1 & \text{if } N \text{ odd} \end{cases}$$

$$E_0 = \begin{cases} \frac{N^2}{4}\hbar\,\omega & \text{if } N \text{ even} \\ \left(\frac{(N-1)^2}{4} + \frac{N}{2}\right)\hbar\,\omega & \text{if } N \text{ odd} \end{cases}$$

N even:

$$E_f = \left(\frac{N}{2} - \frac{1}{2}\right)\hbar\,\omega$$

N odd:

$$E_f = \frac{N}{2}$$

3.20 Add $j_1=1$, $j_2=1$ to form $j=2,1,0$ states $|j,m\rangle$ in terms of $|j_1,j_2,m_1,m_2\rangle$

$$|j_1-j_2|\leqslant j\leqslant j_1+j_2,\quad m=m_1+m_2;\quad \hbar=1$$

$$|++\rangle,\quad |+0\rangle,\quad |0+\rangle,\quad |0,0\rangle,\quad |0-\rangle,\quad |-+\rangle,\quad |-0\rangle,\quad |--\rangle$$

$$|j=2,m=2\rangle \quad = \quad |++\rangle$$

$$J_\pm|j,m\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle;\quad \boldsymbol{J}=\boldsymbol{J}_1+\boldsymbol{J}_2$$

$$J_-|2,2\rangle=(J_{-1}\oplus J_{2-})\,|++\rangle\propto\{|0,+\rangle+|+,0\rangle\}$$

$$|2,1\rangle=\frac{1}{\sqrt{2}}\,(|0+\rangle+|+0\rangle)$$

$$J_-|2,1\rangle=(J_{1-}\oplus J_{2-})\,\frac{1}{\sqrt{2}}(|0,+\rangle+|+,0\rangle)\propto|-,+\rangle+|0,0\rangle+|0,0\rangle+|+,-\rangle$$

$$|2,0\rangle=\frac{1}{\sqrt{6}}(|-,+\rangle+2|0,0\rangle+|+,-\rangle)$$

$$J_-|2,0\rangle\propto|-,0\rangle+2(|-,0\rangle+|0,-\rangle)+|0,-\rangle$$

$$J_-|2,-1\rangle\propto|-,-\rangle=|2,m=-2\rangle$$

$j=1$: $|1,1\rangle,|1,0\rangle,|1,-1\rangle$

$$|1,1\rangle=a\,|+,0\rangle+b|0,+\rangle$$

$$\langle 2,1|1,1\rangle=0\quad\Rightarrow\quad |1,1\rangle=\frac{1}{\sqrt{2}}(|+,0\rangle-|0,+\rangle)$$

$$J_+|1,1\rangle\propto|+,+\rangle-|+,+\rangle=0$$

$$J_-|1,1\rangle\propto|0,0\rangle+|+,-\rangle-|-,+\rangle-|0,0\rangle=|+,-\rangle-|-,+\rangle$$

$$|1,-1\rangle=\frac{1}{\sqrt{2}}\,(|0,-\rangle-|-,0\rangle)$$

$$|0,0\rangle=a\,|+,-\rangle+b|0,0\rangle+c|-,+\rangle$$

$$\langle 0,0|0,0\rangle=1$$

$$\langle 0,0|1,0\rangle=0$$

$$\langle 0,0|2,0\rangle=0$$

$$|0,0\rangle=\frac{1}{\sqrt{3}}\,(|+,-\rangle-|0,0\rangle+|-,+\rangle)$$

6.5 Consider wealky interacting bosonic particles.

a) Suppose the space part of the state vector is symmetric. Construct the normalised spin states in the following thre cases:

1. All three in $|+\rangle$.

2. Two in $|+\rangle$ and on in $|0\rangle$.

3. All thre different.

1. $|+\rangle|+\rangle|+\rangle$. $j=3$

2. $\frac{1}{\sqrt{3}}(|+\rangle|+\rangle|0\rangle+|+\rangle|0\rangle|+\rangle+|0\rangle|+\rangle|+\rangle)$. $j=3$. $J_-|+++\rangle=...$

3. $\frac{1}{\sqrt{6}}(|+\rangle|0\rangle|-\rangle\ +\ |0\rangle|+\rangle|-\rangle\ +\ |+\rangle|-\rangle|0\rangle\ +\ |-\rangle|0\rangle|+\rangle\ +\ |0\rangle|-\rangle|+\rangle\ +\ |-\rangle+\rangle|0\rangle)$.

$$\boldsymbol{J}=\boldsymbol{J}_A+\boldsymbol{J}_B+\boldsymbol{J}_C$$

b) Same as in a, but the space part is totally antisymmetric.

1. —

2. —

3.

$$\frac{1}{\sqrt{6}}\{|+0-\rangle-|+-0\rangle+|0-+\rangle-|0+-\rangle+|-+0\rangle-|-0+\rangle\}$$

$m=0$, $(j=0)$

$|\alpha^{(i)}\rangle$