# Övning 2006–03–29

**Lay 6.8.1** Viktad minsta-kvadrat. Anpassa $y = \beta_0 + \beta_1 t$ till $\begin{array}{c|ccccc} t & -2 & -1 & 0 & 1 & 2 \\ \hline y & 0 & 0 & 2 & 4 & 4 \end{array}$. Vi sätter hälften så stor vikt vid ändpunkterna.

Överbestämt ekvationssystem med vikter:

$$\left\{\begin{array}{l} \frac{1}{2}\cdot(\beta_0 - 2\cdot\beta_1) = \frac{1}{2}\cdot 0 \\ 1\cdot(\beta_0 - \beta_1) = 1\cdot 0 \\ 1\cdot(\beta_0 + 0\cdot\beta_2) = 1\cdot 2 \\ 1\cdot(\beta_0 + 1\cdot\beta_1) = 1\cdot 4 \\ \frac{1}{2}\cdot(\beta_0 + 2\cdot b_1) = \frac{1}{2}\cdot 4 \end{array}\right. \Longleftrightarrow A\boldsymbol{x} = \boldsymbol{b}$$

$$A = \begin{pmatrix} \frac{1}{2} & & & & \mathbf{0} \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \mathbf{0} & & & & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} \frac{1}{2} & & & & \mathbf{0} \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \mathbf{0} & & & & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{pmatrix}$$

Normalekvationerna $A^T A \left(\begin{smallmatrix} \beta_0 \\ \beta_1 \end{smallmatrix}\right) = A^T\boldsymbol{b}$ blir:

$$\begin{pmatrix} 3.5 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} \Longrightarrow \left\{\begin{array}{l} \beta_0 = 2 \\ \beta_1 = \frac{3}{2} \end{array}\right.$$

**Heath 3.3** Anpassa $f(x,t) = x_1\cdot t + x_2\cdot \mathrm{e}^t$ till tabellen $\begin{array}{c|ccc} t & 1 & 2 & 3 \\ \hline f & 2 & 3 & 5 \end{array}$.

Vårt överbestämda system:

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} 1 & \mathrm{e}^1 \\ 2 & \mathrm{e}^2 \\ 3 & \mathrm{e}^3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \equiv \boldsymbol{b}$$

Normalekvationerna:

$$A^T A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^T\boldsymbol{b} \Longrightarrow \begin{pmatrix} 14 & 77.753 \\ 77.753 & 465.416 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 23 \\ 128.0314 \end{pmatrix} \Longrightarrow \left\{\begin{array}{l} x_1 = 1.5942 \\ x_2 = 0.0088 \end{array}\right.$$

**Extra 10** Anpassa $I(t) = I_0\cdot \mathrm{e}^{-\lambda t}$, $I_0$, $\lambda$ är obekanta till given tabell. Detta var inte någon linjär relation mellan de obekanta. Att köra icke-linjär minsta-kvadrat kan vara besvärligt; man försöker ofta linearisera problemet. I detta exempel innebär detta logaritmering.

Anpassa $\ln(I(t))$ till den "loggade" tabellen; $\ln(I(t)) = \ln\left[I_0\cdot \mathrm{e}^{-\lambda t}\right] = \ln(I_0) - \lambda t = a - \lambda t$.

| $t$ | 1 | 2 | 3 | $\cdots$ | 6 |
|---|---|---|---|---|---|
| $\ln(I)$ | $\ln(6.32)$ | $\ln(4.76)$ | $\cdots$ | $\cdots$ | $\ln(1.48)$ |

$$A^T A\begin{pmatrix} a \\ \lambda \end{pmatrix} = A^T\boldsymbol{b} \Longrightarrow \left\{\begin{array}{l} a = 2.1338 \quad\Longrightarrow\quad I_0 = \mathrm{e}^a = 8.45 \\ \lambda = 0.289 \end{array}\right.$$

**Lay 8.5** Vårt rum är $V=\mathcal{C}^0[0,2\pi]$ med skalärprodukt $\langle f,g\rangle=\int_0^{2\pi} f(t)g(t)\mathrm{d}t$.

Påstående: för $m\in\mathbb{N}, n\in\mathbb{N}, m\neq n$ gäller $\sin(m\,t)\perp\sin(n\,t)$. Kollar

$$\int_0^{2\pi}\sin(m\,t)\sin(n\,t)\,\mathrm{d}t$$

$$\begin{cases}\cos(m\,t+n\,t)=\cos(m\,t)\cos(n\,t)-\sin(m\,t)\sin(n\,t)\\ \cos(m\,t-n\,t)=\cos(m\,t)\cos(n\,t)+\sin(m\,t)\sin(n\,t)\end{cases}$$

$$\Longrightarrow 2\sin(m\,t)\sin(n\,t)=\cos(m\,t-n\,t)-\cos(m\,t+n\,t)$$

$$\int_0^{2\pi}\sin(m\,t)\sin(n\,t)\,\mathrm{d}t=\ldots=0$$

**Lay 6.8.9** Sök tredje ordningens fourierutveckling av $f(t)=2\pi-t$ på $[0,2\pi]$; skalärprodukt:

$$\langle h,g\rangle=\int_0^{2\pi}h(t)g(t)\,\mathrm{d}t$$

Det vill säga:

$$f(t)\approx\frac{a_0}{2}+a_1\cos(1\cdot t)+b_1\sin(1\cdot t)+a_2\cos(2\cdot t)+b_2\sin(2\cdot t)+a_3\cos(3\cdot t)+b_3\sin(3\cdot t)$$

$$\frac{a_0}{2}=\frac{1}{2}\cdot\frac{1}{\pi}\cdot\int_0^{2\pi}f(t)\,\mathrm{d}t=\frac{1}{2\pi}\int_0^{2\pi}(2\pi-t)\,\mathrm{d}t=\pi$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos(k\,t)\,\mathrm{d}t=0$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin(k\,t)=\ldots=0+\frac{2}{k}$$

Vi får:

$$\pi+0+2\sin(t)+0+\sin(2\,t)+0+\frac{2}{3}\sin(3\,t)$$

**Extra 28** Bestäm en ON-bas till $U^T$, $U$:s ortogonala komplement, med

$$U=\mathrm{Span}\left\{\begin{pmatrix}1\\2\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\3\\0\\0\end{pmatrix}\right\}$$

Om $\boldsymbol{x}\in U^T\Longrightarrow\boldsymbol{x}\perp\boldsymbol{u}\forall\boldsymbol{u}\in U\Longrightarrow\boldsymbol{x}\perp$mot de båda vektorer som spänner $U$ (Anmärkning: dessa vektorer är linjärt oberoende: bas i $U$). Med

$$A=\begin{pmatrix}1&2&0&0&0\\1&0&3&0&0\end{pmatrix}\Longrightarrow A\,\boldsymbol{x}=\begin{pmatrix}0\\0\end{pmatrix}\Longrightarrow\boldsymbol{x}\in\mathrm{Nul}(A)$$

Reducera, Gauss:

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{x} = t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dessa var linjärt oberoende, till och med ortogonala. Normera dem:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{7} \begin{pmatrix} -6 \\ 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

**Heath 3.14** Med $\boldsymbol{v} \neq \boldsymbol{0}$ bilda

$$H = I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}$$

Påstående: $H$ är en ortogonal matris: $H^T H = I$.

$$H^T H = \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right)^T \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) = \left(I - 2 \frac{(\boldsymbol{v}\, \boldsymbol{v}^T)^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) =$$

$$= \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) = I$$

$H^T = H \Longrightarrow H$ är symmetrisk.

**Heath 3.15** Tag $\boldsymbol{a} \neq \boldsymbol{0}$, bilda $\boldsymbol{v} = \boldsymbol{a} - \alpha \boldsymbol{e}_1; \alpha = \pm \|\boldsymbol{a}\|_2$. Tag nu

$$H = \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) \Longrightarrow H\, \boldsymbol{a} = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \alpha \boldsymbol{e}_1$$

$$A = [\boldsymbol{a} \quad \boldsymbol{a}_2 \quad ...]$$

$$H\, A = \begin{pmatrix} \alpha & // & // \\ 0 & // & // \\ \vdots & // & // \\ 0 & // & // \end{pmatrix}$$

$$H\, \boldsymbol{a} = \left(I - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T}{\boldsymbol{v}^T \boldsymbol{v}}\right) \boldsymbol{a} = \boldsymbol{a} - 2\, \frac{\boldsymbol{v}\, \boldsymbol{v}^T \boldsymbol{a}}{\boldsymbol{v}^T \boldsymbol{v}} = \boldsymbol{a} - \frac{2(\boldsymbol{v}^T \boldsymbol{a})}{\boldsymbol{v}^T \boldsymbol{v}} = \boldsymbol{a} - 2 \frac{\boldsymbol{v}^T \boldsymbol{a}}{\boldsymbol{v}^T \boldsymbol{v}}\, \boldsymbol{v}$$

Betrakta konstanten

$$2\,\frac{\boldsymbol{v}^T\boldsymbol{a}}{\boldsymbol{v}^T\boldsymbol{v}} = 2\,\frac{(\boldsymbol{a}-\alpha\boldsymbol{e}_1)^T\boldsymbol{a}}{(\boldsymbol{a}-\alpha\boldsymbol{e}_1)^T(\boldsymbol{a}-\alpha\,\boldsymbol{e}_1)} = 2\,\frac{\boldsymbol{a}^T\boldsymbol{a}-\alpha\boldsymbol{e}_1^T\boldsymbol{a}}{\boldsymbol{a}^T\boldsymbol{a}-\alpha\boldsymbol{a}^T\boldsymbol{e}_1-\alpha\,\boldsymbol{e}_1^T\boldsymbol{a}+\alpha^2\,\boldsymbol{e}_1^T\boldsymbol{e}_1} =$$

$$[\boldsymbol{a}^T\boldsymbol{a} = \|\boldsymbol{a}\|_2^2 = \alpha^2] = 2\,\frac{\alpha^2-\alpha\,\boldsymbol{e}_1^T\boldsymbol{a}}{2\,\alpha^2-2\,\alpha\,\boldsymbol{e}_1^T\boldsymbol{a}} = 1$$

$$H\,\boldsymbol{a} = \boldsymbol{a}-\boldsymbol{v} = \boldsymbol{a}-(\boldsymbol{a}-\alpha\,\boldsymbol{e}_1) = \alpha\,\boldsymbol{e}_1$$

**Heath 3.22** Med $A\in\mathbb{R}^{m\times n}; m\geqslant n; \operatorname{rang}(A)=n$.

Med

$$A = Q\,R = Q\left(\begin{array}{c} \begin{matrix} \cdot & \cdot & \cdot \\ & \cdot & \cdot \\ \mathbf{0} & & \cdot \end{matrix} \\ \hline \\ \mathbf{0} \\ \\ \end{array}\right) = Q\begin{pmatrix} R_1 \\ \mathbf{0} \end{pmatrix}$$

där $R\in\mathbb{R}^{n\times n}$ är triangulär.

$A^TA$, symmetrisk, kan faktoriseras $A^TA = L\,L^T \Longrightarrow R_1^TR_1 = L\,L^T$.

$$A^T A = L\,L^T = (Q\,R)^T(Q\,R) = \left(Q\begin{pmatrix} R_1 \\ \mathbf{0} \end{pmatrix}\right)^T\left(Q\begin{pmatrix} R_1 \\ \mathbf{0} \end{pmatrix}\right) = \begin{pmatrix} R_1 \\ \mathbf{0} \end{pmatrix}^T Q^TQ\begin{pmatrix} R_1 \\ \mathbf{0} \end{pmatrix} = R_1^TR_1$$

Entidighet om dianonalelementen i $R>0$.