komplex analys

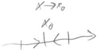
Analytiska funktioner

$f: \mathbb{C} \to \mathbb{C}$

analytisk funktion $\approx$ deriverbar

Hur z än går mot z_0

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$



$D \subset \mathbb{C}$

Område (domain) { sammanhängande, öppen mängd }

f deriverbar i $D \Rightarrow f \in C^{\infty}(D)$
(analytisk)

$f|_\gamma$ känd $\Rightarrow$ f innanför γ kan rekonstrueras

$f|_\gamma = 0 \Rightarrow f \equiv 0$ i D

Komplexa tal

$\mathbb{N} = \{1, 2, 3, \dots\}$ basbegrepp (defineras ej)

Peanos axiomatik

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

$-1 = "1-2"$

$\mathbb{N}$: $+$, $-$? $\quad 5-3=2$, $3-5$?

def: $\mathbb{Z} = \{(m,n) : m, n \in \mathbb{N}\}$

(i bakhuvudet: $m-n$
$(3,2) = 3-2 = 4-3 = (4,3)$)

där $(m,n) = (p,q)$ om $m+q = n+p$

$\rightarrow$

② $\mathbb{C}$?

$x^2+1=0$ har ej lösningar i $\mathbb{R}$

Önskemål: 1) $\mathbb{R} \subset \mathbb{C}$

2) i gärligaste mån bevara räknereglerna

3) $x^2+1=0$ lösbar

$\mathbb{C} \overset{\text{def}}{=} \{(a,b) : a,b \in \mathbb{R}\}$

$(a,b) = (c,d)$ omm $\begin{matrix} a=c \\ b=d \end{matrix}$ (i beteckningen: $a+bi$)

def $(a,b)+(c,d) = (a+c, b+d)$

$(a,b) \cdot (c,d) = (ac-bd, bc+ad)$

kommutativitet, associativitet, distributivitet

1. $\mathbb{R} \subset \mathbb{C}$

$\tilde{\mathbb{R}} = \{(a,0) : a \in \mathbb{R}\} \subset \mathbb{C}$

$\mathbb{R} \cong \tilde{\mathbb{R}}$ isomorfism

$\mathbb{R}, \tilde{\mathbb{R}}$ är isomorfa

$a \text{ "="} (a,0)$ $\quad (a,0)+(c,0) = (a+c, \underbrace{0+0}_{=0}) \text{ "="} a+c$

$(a,0)\cdot(c,0) = (ac - 0\cdot 0, 0\cdot c + a\cdot 0) \text{ "="} ac$

$\Rightarrow \mathbb{R} \cong \tilde{\mathbb{R}} \subset \mathbb{C}$

2. $x^2+1=0$ lösbar

$(0,1)^2 + (1,0) = \underbrace{(0\cdot 0 - 1\cdot 1, 0\cdot 1 + 1\cdot 0)}_{(-1,0)} + (1,0) = (0,0) = 0$

$\Rightarrow x^2+1=0$ lösbar i $\mathbb{C}$

def $i = (0,1)$ den imaginära enheten

Algebraisk form: $(a,0)+(b,0)(0,1) \overset{\text{visa!}}{=} (a,b)$

$a+bi = (a,b)$

$i = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = (-i)i = -i^2 = 1$
$i^5 = i$
$\vdots$

Algebraisk form: $z = a + bi \,(= (a,b))$

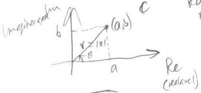
Division: $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di} = \dfrac{ac - adi + bci - bdi^2}{c^2 - (di)^2} = \dfrac{ac + bd + (bc-ad)i}{c^2+d^2}$ >0 Om $(c,d) \neq (0,0)$

$= \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}\, i$

def $z = a + bi$
$\bar{z} = a - bi$ $\quad \bar{z}$: s komplexkonjugat

$z\bar{z} = a^2 + b^2$ $\quad a = \operatorname{Re} z$ $\quad z$:s realdel
$\quad b = \operatorname{Im} z$ $\quad z$:s imaginärdel $\in \mathbb{R}$
OBS!

Geometrisk tolkning
$z = (a,b)$ $\quad a, b \in \mathbb{R}$
$\mathbb{C}$ $\quad$ komplexa talplanet



$(r =)\ |z| = \sqrt{a^2+b^2}$

Polär form
$\begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$

Givet r, θ: lätt att entydigt få a, b $(a + bi)$

Givet a, b: entydigt bestämma $r = |z| = \sqrt{a^2+b^2}$
$\arg z$ = ej entydigt bestämt
argument
Om θ_0 är ett möjligt värde för $\arg z$, så $\arg z = \theta_0 + 2k\pi$, $k \in \mathbb{Z}$
flertydig funktion
0 har ingen polär form $(|0| = 0$, saknar argument$)$

$\rightarrow$

④ Polar form: $z = r(\cos\theta + i\sin\theta)$

$z_{1,2} = r_{1,2}(\cos\theta_{1,2} + i\sin\theta_{1,2})$.

$$z_1 z_2 = r_1 r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\big)$$
$$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

Specialfall: $|z_1| = |z_2| = 1$

$$z_1 z_2 = \underbrace{(\cos\theta_1 + i\sin\theta_1)}_{f(\theta_1)}\underbrace{(\cos\theta_2 + i\sin\theta_2)}_{f(\theta_2)} = \underbrace{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)}_{f(\theta_1+\theta_2)}$$

$(e^{x_1}e^{x_2} = e^{x_1+x_2})$

def $e^{i\theta} \overset{def}{=} \cos\theta + i\sin\theta$

def $e^{z} = e^{x+iy} \overset{def}{=} e^{x}(\cos y + i\sin y)$

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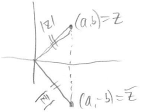
z $\quad \bar{z}$

$|z|$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$



$|z_1 z_2| = |z_1||z_2|$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

$|\bar{z}| = |z|$

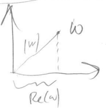
$\big||z_1| - |z_2|\big| \overset{②}{\le} |z_1 + z_2| \overset{①}{\le} |z_1| + |z_2|$ (triangelolikheten)

Bevis ① V.l, h.l $\geq 0 \Rightarrow$ kvadrering ger ekvivalent olikhet

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= z_1\bar{z}_1 + \underline{z_1\bar{z}_2 + z_2\bar{z}_1} + z_2\bar{z}_2 =$$
$$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2 =$$

$(\bar{\bar{z}} = z)$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2 \leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 = |z_1|^2 + 2|z_1||z_2| + |z_2|^2$



$|w| \geq Re\, w$

$"=" \Leftrightarrow w \geq 0$ icke-negativt reellt tal

$= (|z_1| + |z_2|)^2$

$"="$; ① ?

Önskelistan: ②"; "gärligaste mån"

Komplexa tal kan inte jämföras! (inga olikheter mellan komplexa tal)

$Re(w) \leq |w|$

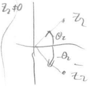
$"=" \Leftrightarrow w \in \mathbb{R}, w \geq 0$

$"=" \Leftrightarrow Re(z_1\bar{z}_2) = |z_1||z_2|$

$\Leftrightarrow (z_1\bar{z}_2 \in \mathbb{R}), z_1\bar{z}_2 \geq 0$

1) $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $\quad z_1 \neq 0$ och $z_2 \neq 0$

$\bar{z}_2 = r_2(\cos(-\theta_2) + i\sin(-\theta_2))$



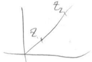
$z_1\bar{z}_2 = r_1 r_2(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \in \mathbb{R}$

$Im(\ldots) = 0 \Leftrightarrow \sin(\theta_1-\theta_2) = 0 \Leftrightarrow \theta_1 - \theta_2 = k\pi, k \in \mathbb{Z}$

$z_1\bar{z}_2 \geq 0$: $\underbrace{\cos k\pi}_{=(-1)^k} \geq 0 \Leftrightarrow k = 2m, m \in \mathbb{Z}$

$"=" \Leftrightarrow \theta_1 = \theta_2 + 2m\pi, m \in \mathbb{Z}$ $\quad \Leftrightarrow \arg z_1 = \arg z_2$

$\Leftrightarrow z_2 = \alpha z_1$, där $\alpha > 0$



2) $z_1 = 0$ eller $z_2 = 0$

$0 = 0$ uppenbart

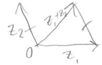
$0 \geq 0$

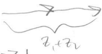
$"=" \Leftrightarrow z_2 = \lambda z_1$

$\lambda \geq 0$

$\rightarrow$

(6)





(7) $|z_1| = |z_1+z_2-z_2| = |(z_1+z_2)+(-z_2)| \overset{(1)}{\le} |z_1+z_2|+|z_2|$

$\Rightarrow |z_1|-|z_2| \le |z_1+z_2|$

$|z_2| = \dots$

$\Rightarrow |z_2|-|z_1| \le |z_1+z_2|$

$$||z_1|-|z_2|| = \begin{cases} |z_1|-|z_2| \dots \\ |z_2|-|z_1| \dots \end{cases} \le |z_1+z_2|$$

$|A+B| \le |A|+|B|$

$$\frac{1}{|A+B|} \le \frac{1}{||A|-|B||}$$

En enda ∞: plats, oändligt långt borta från 0 oavsett i vilken riktning man går.

$z \to \infty \Leftrightarrow |z| \to +\infty$

$\mathbb{R}$ ordnad $a,b \in \mathbb{R}$ $\begin{cases} a=b \\ a<b \\ a>b \end{cases}$ vissa egenskaper

$-\infty$ ——— ∞

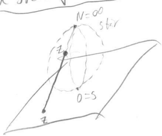
$\mathbb{C}$: ingen ordning

∞

Övning: Antag att det går att definiera >0 i $\mathbb{C}$ med de egenskaper som gäller i $\mathbb{R}$. Kom fram till motsägelse genom att använda i.

Inga olikheter mellan komplexa tal!

Den stereografiska projektionen



$f : \underset{\text{område}}{D} \to \mathbb{C}$

1,1,11 $\quad z^2 \overset{?}{=} (-z)^2$ m.h.a. den formella talpardefinitionen

$z = (a,b) \qquad zw = (ac-bd,\ ad+bc)$

$w = (c,d) \qquad z^2 = (a^2-b^2,\ ab+ba) = (a^2-b^2,\ 2ab) \left(= a^2-b^2+2abi \right)$

$-z$ def $\quad z + (-z) = (0,0) \Rightarrow -z = (-a,-b)$

0 def $\quad w + 0 = w \ \forall w$

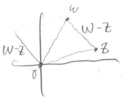
$(a,b)+(c,d) = (a+c,\ b+d) \Rightarrow 0_{\mathbb{C}} = (0,0)$

$(-a,-b)(-a,-b) = \left((-a)^2 - (-b)^2,\ (-a)(-b)+(-b)(-a) \right) = z^2$

reella, byt − mot +

1,1,15 $\quad \Delta$ med horn $0, z, w$

Liksidig $\Leftrightarrow |z|^2 = |w|^2 = 2\mathrm{Re}\,(z\bar{w})$



$\boxed{\Rightarrow}$ Givet Δ liksidig

$|z| = |w| = |w-z|$

$\Leftrightarrow |z|^2 = |w|^2$

$|w-z|^2 = (w-z)\overline{(w-z)} = (w-z)(\bar{w}-\bar{z}) =$

$= w\bar{w} - (w\bar{z} + z\bar{w}) + z\bar{z} = |w|^2 - 2\mathrm{Re}\, w\bar{z} + |z|^2 \overset{\text{Givet}}{=} |z|^2 = |w|^2$

$\Rightarrow |w|^2 = 2\mathrm{Re}\, w\bar{z} = 2\mathrm{Re}\, z\bar{w}$

$\Rightarrow$

(8)

$\boxed{\Leftarrow}$ $|w|^2 = 2\operatorname{Re} z\bar{w}$

$|w|^2 - 2\operatorname{Re} z\bar{w} = 0$

$+|z|^2 \quad |z|^2$

$= |w-z|^2 = |z|^2 = |w|^2 \Rightarrow \Delta$ liksidig

1.2.5 Bestäm mängden $\{z \in \mathbb{C} : |z+2| + |z-2| = 5\}$

$z-(-2)$

$\Leftrightarrow (|z+2| + |z-2|)^2 = 25$

$\underbrace{(z+2)(\bar{z}+2)} + 2|z^2-4| + (z-2)(\bar{z}-2) = 25$

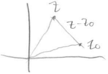
$|z|^2 + 2z + 2\bar{z} + 4$

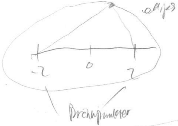
$= |z|^2 + 2\operatorname{Re} z + 4 =$

$x+iy \quad = x^2+y^2+2x+4$

$|z-z_0|$ = avståndet mellan z och z_0

$\sqrt{(x-x_0)^2+(y-y_0)^2}$





1.2.19 $p > 0$ $\quad \Gamma = \{z ; |z-p| = cx, \ z = x+iy\}$

(1) Γ ellips för $0 < c < 1$ $\qquad c = 0 : \Gamma = \{p\}$

(2) parabel $\quad c = 1$

(3) hyperbel $\quad c > 1$

$\Gamma = ?, \ c < 0$

~~$\Gamma = \emptyset$~~ $\quad x < 0$

$|z-p|^2 = c^2x^2$

$= (z-p)(\bar{z}-p) = |z|^2 - p(z+\bar{z}) + p^2$

$x^2+y^2-2px+p^2 = c^2x^2$

$(1-c^2)x^2+y^2-2px+p^2=0$

OBS! Inga xy-termer ⑨

(1) $0<c<1$ $\quad 1-c^2>0$
ellips

(2) $c=1$ parabel

(3) $c>1$ hyperbel

$2x^2+y^2+1=0$ $\quad \emptyset$

kvadrat komplettera!

$\underline{1,2,21}$ $\quad \alpha \in \mathbb{C} \ : \ 0<|\alpha|<1$

(a) $\{z : |z-\alpha| < |1-\bar{\alpha}z|\} = \{z ; |z|<1\}$ enhetsskivan

$\boxed{(b)}$ $\qquad =$ $\qquad \cancel{S_z}$ $\quad |z|=1$ enhetscirkeln

(c) $\qquad >$ $\qquad |z|>1$

$|z-\alpha|^2 = |1-\bar{\alpha}z|^2 \qquad (z-\alpha)(\bar{z}-\bar{\alpha}) = (1-\bar{\alpha}z)(1-\alpha\bar{z})$

$\Leftrightarrow |z|^2 - \cancel{2\mathrm{Re}\,\bar{\alpha}z} + |\alpha|^2 = 1 - \cancel{2\mathrm{Re}\,\bar{\alpha}z} + |\alpha|^2|z|^2$

$\Leftrightarrow \underbrace{(1-|\alpha|^2)}_{\neq 0 \text{ ty } 0<|\alpha|<1}|z|^2 = 1-|\alpha|^2$

$\Leftrightarrow |z|^2 = 1 \Leftrightarrow |z|=1$

$z_0 = \alpha$

Sammanh.

Sammanh.

$f(z) = \left|\dfrac{z-\alpha}{1-\bar{\alpha}z}\right|$ kontinuerlig (Visa nämnaren $\neq 0$ där den inte får bli 0)

$f \equiv 1$ på $\{|z|=1\}$

f enbart $=1$ på

Välj $z_0 = \alpha \in \{|z|<1\}$

$f(z_0) = f(\alpha) = \left|\dfrac{0}{1-|\alpha|^2}\right| = 0$

p.g.a. kontinuiteten $f(z)<1$ i $\{|z|<1\}$

Välj $z_1 = \frac{1}{\bar{\alpha}} \in \{|z|>1\}$

$f(z_1) = f\left(\frac{1}{\bar{\alpha}}\right) = \left|\dfrac{\frac{1}{\bar{\alpha}} - \alpha}{1-\frac{\bar{\alpha}}{\bar{\alpha}}}\right| = \dfrac{\cdots \neq 0}{0} = |\infty| > 1$

$\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ $\qquad$ (låt $z \to \frac{1}{\bar{\alpha}}$, $f(z) \to \infty$ >1)

alternativt gör som i (b), d.v.s. utveckla.

$\rightarrow$

Komplexa funktioner

$D \subset \mathbb{C}$ D område

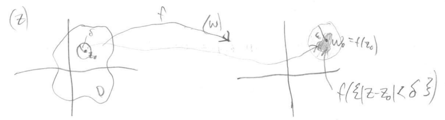
$f: D \to \mathbb{C}$

<u>def</u> $z_0 \in D$ (OBS! D öppen $\Rightarrow z_0$ inre pkt)

f kallas kontinuerlig i z_0 om

$$\exists \lim_{z \to z_0} f(z) = f(z_0),$$

d.v.s. $\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 : \forall z \in D : |z - z_0| < \delta \; |f(z) - f(z_0)| < \varepsilon$



<u>def</u> $z_0 \in D$

f deriverbar i z_0 om $\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

I så fall: $\lim \ldots = f'(z_0)$
f:s derivata i z_0

<u>def</u> f kontinuerlig/deriverbar i D om f kontinuerlig/deriverbar i z $\forall z \in D$

Antag att f är deriverbar i $z_0 \in D$

$z = x + iy \quad x, y \in \mathbb{R}$

$z_0 = x_0 + iy_0$

$f(z) = \mathrm{Re} f + i \mathrm{Im} f$

$\mathrm{Re} f = u(x,y) \quad f(z) = u(x,y) + iv(x,y)$

$\mathrm{Im} f = v(x,y)$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad (z = z_0 + \Delta z)$$

Låt $\Delta z = h \in \mathbb{R}$ ($z \to z_0$ parallellt med realaxeln)

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{u(x_0+h,y_0)+iv(x_0+h,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h} =$$

$$= \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + i\,\frac{v(x_0+h,y_0)-v(x_0,y_0)}{h} \xrightarrow[h\to 0]{} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z_0)$$

i punkten (x_0,y_0)

(har differenskvoterna för u,v gränsvärden?)

Låt $\Delta z = ik$, $k \in \mathbb{R}$ ($z \to z_0$ parallellt med imaginäraxeln) $z_0+ik = x_0+i(y_0+k)$

$$\frac{f(z_0+ik)-f(z_0)}{\underbrace{ik}_{=\Delta z}} = \frac{u(x_0,y_0+k)-u(x_0,y_0)}{ik} + \cancel{i}\,\frac{v(x_0,y_0+k)-v(x_0,y_0)}{\cancel{i}k} \to \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} =$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f'(z_0) \implies f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

i (x_0,y_0)

$$\implies \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & \text{i } (x_0,y_0) \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} & \text{i } (x_0,y_0) \end{cases}$$

Cauchy-Riemanns ekvationer (C-R)

f deriverbar i $D \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ $\quad \forall z = x+iy \in D$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} & \text{i } (x,y) \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} & \text{-}''\text{-} \end{cases}$$

f deriverbar i $D \Rightarrow f(u,v)$ uppfyller C-R i D.

Exempel $f(z) = \bar{z} = x - iy$ $\quad u(x,y) = x$, $v(x,y) = -y$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

$\Rightarrow f$ ej deriverbar i ngn punkt.

(12)

Är C-R tillräckligt villkor för deriverbarhet?
(kan man vända pilen?)

Svar: Nja...

Sats $u, v \in C^1(D)$
och uppfyller C-R-ekvationer
$\Rightarrow f = u + iv$ är deriverbar i D

---

def f kallas analytisk (holomorf) i z_0 om f är deriverbar i en omgivning till z_0.

Elementära funktioner

potenser, exponentialfunktioner, trigonometriska funktioner, deras inverser, ändliga sammansättningar av sådana

$\Gamma(z)$ ej elementär
$|x|$

$\left(\int e^{\pm x^2} dx \quad \int \frac{dx}{\ln x} \quad \text{ej elementära} \quad \int \frac{\sin x}{x} dx \right)$

Potenser: $z^n \overset{\text{def}}{=} \underbrace{z \cdot z \cdots z}_{n\ \text{st.}}$
$n \in N$

$z^0 \overset{\text{def}}{=} 1 \quad (z \neq 0)$

$z^{-n} = \frac{1}{z^n} \quad (z \neq 0)$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}$

$z \neq 0 \begin{cases} z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta)) \\ z^0 = 1 = r^0(\cos 0 + i\sin 0) \end{cases}$

$\Rightarrow z^m = r^m(\cos m\theta + i\sin m\theta) \quad \forall m \in \mathbb{Z} \quad$ (ev. om $z \neq 0$)

$z^\alpha = ?$ för $\alpha \in \mathbb{C}$ $\quad (x^\alpha = e^{\alpha \ln x}$, $x > 0)$

$\Rightarrow$

z^n analytisk, ty $\frac{u}{v}$ uttrycks i potenser av x och y
$(z \neq 0)$

---

det $e^z = e^x(\cos y + i \sin y)$ för $z = x+iy$ $D = \mathbb{C}$

$e^{z_1+z_2} = e^{x_1+x_2} e^{i(y_1+y_2)} = e^{x_1+x_2} \cdot e^{iy_1} \cdot e^{iy_2} = e^{z_1} e^{z_2}$

$e^0 = e^{\operatorname{Re} 0}(\cos 0 + i \sin 0) = 1$

$|e^z| = e^x > 0 \Rightarrow e^z \neq 0 \quad \forall z \in \mathbb{C}$

$\arg e^z = y + 2k\pi, \ k \in \mathbb{Z}$

e^z periodisk med period $2\pi i$

? $\exists (e^z)' = e^z$

$u = \operatorname{Re} e^z = e^x \cos y \in C^\infty$? C-R

$v = \operatorname{Im} e^z = e^x \sin y \in C^\infty$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y} = e^x(+\sin y)$$

$\Rightarrow e^z$ $\underbrace{\text{analytisk i hela } \mathbb{C}}_{\text{hel funktion}}$

$$(e^z)' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^z$$

$e^{ix} = \cos x \pm i \sin x$ $\quad \cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ $\quad \sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

$x \in \mathbb{R}$

det $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

hela funktioner

$\Rightarrow$

(14)

def $D \subset \mathbb{R}^2$ $\quad u: D \to \mathbb{R},\ u \in C^2(D)$

u kallas harmonisk i D om $\Delta u = (\nabla^2 u =) \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$; D

Laplace

Sats f analytisk i D, $f = u + iv \Rightarrow u, v$ harmoniska i D

Bevis $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Big| \dfrac{\partial}{\partial x}$ $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y}$

(under förutsätt att $u, v \in C^2$, eg. behövs ej)

$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Big| \dfrac{\partial}{\partial y}$ $\quad \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y}$

Addera: $\Delta u =$, analogt för $\underline{v}$

---

def $e^z = e^x(\cos y + i \sin y)$

e^z hel funktion (analytisk i $\mathbb{C}$)

$(e^z)' = e^z$

$\left(f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}\right)$

$\exists f' \overset{?}{\Rightarrow} \exists \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \ldots$ imorgon

---

def $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad$ (om $z = x \in \mathbb{R}$, så får vi de vanliga sin, cos)

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

def $\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$

Samma formler gäller som i reella fallet.

$(\sin z)' = \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)' = \dfrac{\cancel{i}e^{iz} + \cancel{i}e^{-iz}}{2\cancel{i}} = \cos z$ $\quad \begin{pmatrix} (f+g)' = f' + g' \\ (fg)' = f'g + fg' \end{pmatrix}$

e^z periodisk med perioden $2\pi i$; $e^z \neq 0 \ \forall z \in \mathbb{C}$

cosh, sinh periodiska

$\left.\begin{array}{l}\sin z \\ \cos z\end{array}\right\}$ obegränsade för $z \in \mathbb{C}$

$$|\sin(iy)| = \left|\frac{e^{-y}-e^{y}}{2i}\right| \overset{\frac{1}{i}=-i}{=} \left|i\,\frac{e^{y}-e^{-y}}{2}\right| = |i\sinh y| = |\sinh y| \xrightarrow[y\to\infty]{} \infty$$

($\sin(iz)$ analogt)

analogt: $\cos(iz) = \cosh z$

Logaritmfunktionen

(ln pos. reella tal, log i komplexa)

Flertydig!

$\log z$?

Inversen till e^z. $e^z = w$ $\log w$?

$$e^z \overset{\text{def}}{=} e^x(\cos y + i\sin y) = w = \rho(\cos\varphi + i\sin\varphi)$$

$$\Rightarrow \begin{cases} e^x = \rho \iff x = \ln\rho \\ y = \varphi + 2k\pi = \underbrace{\arg w}_{\text{flertydig}} \quad k\in\mathbb{Z} \end{cases}$$

$$z = \log w \text{ om } \begin{cases} x = \operatorname{Re} z = \ln|w| \\ y = \operatorname{Im} z = \arg w \end{cases}$$

$$\boxed{\log z = \ln|z| + i\arg z,\ \forall z \neq 0} \text{ flertydig}$$

$\arg z = \theta_0 + 2k\pi$

$\log z = \ln|z| + i(\theta_0 + 2k\pi)$, $k\in\mathbb{Z}$

ex ① $\log(-1) = \underbrace{\ln|-1|}_{0} + i(\pi + 2k\pi)$

$= i(2k+1)\pi$, $k\in\mathbb{Z}$

② $\log(1+i) = \ln\sqrt{1^2+1^2} + i\left(\frac{\pi}{4} + 2k\pi\right)$

$= \ln\sqrt{2} = \frac{1}{2}\ln 2$

---

def $z^{\alpha} = e^{\alpha\log z}$) i allmänhet flertydig

$z, \alpha \in \mathbb{C}$, $z \neq 0$

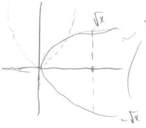
$$z^2 = e^{2\log z} = e^{2(\ln|z| + i(\theta_0 + 2k\pi))} = \text{flertydigt} = |z|^2$$

$$= e^{2\ln|z| + 2i\theta_0} \cdot e^{4ik\pi} = e^{(\ln|z|)^2} e^{i2\theta_0} =$$

($\cos 4k\pi + i\sin 4k\pi = 1$)

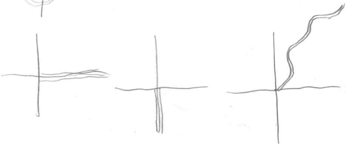
$= z^2$

(16) Envydiga grenar av flertydiga funktioner



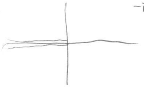
Entydighet: man ska inte kunna gå runt 0

Snitt: mellan 0 och ∞



Principalgren av $\arg z$:

$$-\pi < \operatorname{Arg} z \leq \pi$$



$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

område $\quad -\pi < \operatorname{Arg} z < \pi$

$\log(-1) = i\pi$

$-\frac{\pi}{2} < \arg_{*} z < \frac{3\pi}{2}$

Exempel $\quad i^i = e^{i \log i} = e^{i(\overset{0}{\ln 1} + i(\frac{\pi}{2} + 2k\pi))} = e^{-(\frac{\pi}{2} + 2k\pi)}, \; k \in \mathbb{Z}$

$f = u + iv$ analytisk i $D \Rightarrow u, v$ harmoniska

(C-R) $\quad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$ $\quad u, v$ varandras konjugat

Fråga: Om u harmonisk i D, finns det en analytisk funktion $f: D \to \mathbb{C}$ s.a. $u = \operatorname{Re} f$?

Ja, om D är enkelt sammanhängande

4.1.1a $u(x,y) = x^4 - 6x^2y^2 + y^4$

$D = \mathbb{R}^2 \text{"="} \mathbb{C}$

$$\Delta u \overset{?}{=} 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

p.g.a symmetri $\left.\frac{\partial^2 u}{\partial y^2} = 12y^2 - 12x^2 \right\} \Rightarrow \Delta u = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 4x^3 - 12xy^2$$

$$v = \int (4x^3 - 12xy^2)\,dy + \underbrace{\varphi(x)}_{\text{const m.a.p. } y} = 4x^3y - 4xy^3 + \varphi(x)$$

$$\frac{\partial v}{\partial x} = 12x^2y - 4y^3 + \varphi'(x) = -\frac{\partial u}{\partial y} = -(4y^3 - 12yx^2) \Rightarrow \varphi'(x) \equiv 0$$

(Om man får $\varphi'(x) = xy$, så $\nexists v$)

$$\Rightarrow \varphi(x) \equiv \underline{C \in \mathbb{R}}$$

$\Rightarrow \exists f(z)$: $u = \operatorname{Re} f$

$$\underline{f(z)} = \underbrace{x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3)}_{\text{uttryck i termer av } \underline{\underline{z}}} + iC$$

1) $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$ $\bar{z}$ ska försvinna (pga f är analytisk)

2) Tag $y = 0$ och byt ut x mot z $\Big\}$ 2, 1

$f(z) = z^4 + iC$

(18) Möbiusavbildningar (kap3; Linear fractional transformations)

$f(z)=\frac{az+b}{cz+d}$, $z\in\overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ $a,b,c,d\in\mathbb{C}$

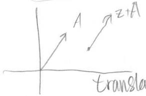
$$\frac{caz\overset{+ad-ad}{+}bc}{acz+da}\cdot\frac{a}{c} = \underbrace{\frac{a}{c}}_{\text{const}}\cdot 1 + \frac{\cancel{a}^{1}}{c}\cdot\frac{bc-ad}{acz+da}$$ krav: $bc-ad\neq 0$

f sammansättning av: $z+A$
Bz
$\frac{1}{z}$

$f(z)$

$$f(z)=\frac{az+b}{cz+d}=\frac{a}{c}+\frac{1}{c}\cdot\frac{bc-ad}{cz+d}$$

$z+A$



translation

$B=\rho e^{i\varphi}$

ρz omskalning

$e^{i\varphi}re^{i\theta} = re^{i(\theta+\varphi)}$

$z=re^{i\theta}$

Rotation

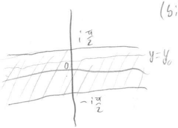
"med "cirklar" menas cirklar eller räta linjer"

(1.5.23) $f(z)=e^z$ $\{z=x+iy: -\infty<x<\infty, -\frac{\pi}{2}<y<\frac{\pi}{2}\} \xrightarrow{f} ?$

entydig (bijektion)

rand $\xrightarrow{f}$ rand
utom? →0!

$y=\pm\frac{\pi}{2}$; $y=y_0$



$z=x+iy$

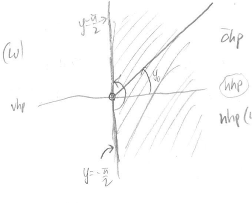
$e^z=e^x(\cos y+i\sin y)$

$|e^z|=e^x$

$0<e^x<\infty$

$-\frac{\pi}{2}<y<\frac{\pi}{2}$ (en) den polära vinkeln för e^z

$\underline{\operatorname{Arg} e^z=y}$



$\operatorname{Log} w=z$

(2.1.16) $\left(\frac{az+b}{cz+d}\right)'=\frac{a(cz+d)-c(az+b)}{(cz+d)^2}=\frac{ad-bc}{(cz+d)^2}$ $\not\equiv 0$ för $ad-bc\neq 0$; $\equiv 0$ $=0$

$a,b,c,d\in\mathbb{C}$

$\not\equiv$ const $\Leftrightarrow ad-bc\neq 0$

(2.1.20 d) $u(x,y)=\cosh y\sin x$

Finn det (ett) harmoniska konjugatet

(v: $f=u+iv$ analytisk)

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$$

$\frac{\partial u}{\partial x}=\boxed{\cosh y\cos x=\frac{\partial v}{\partial y}}$ $v(x,y)=\int\cosh y\cos x\,dy+\varphi(x)=$

$=\sinh y\cos x+\varphi(x)$

$\frac{\partial v}{\partial x}=-\sinh y\sin x+\varphi'(x)=-\sinh y\sin x$

$\Rightarrow\varphi'(x)=0\Rightarrow\varphi\equiv C\in\mathbb{R}$

$f(z)=\cosh y\sin x+i(\sinh y\cos x+C)$ $\cos(iy)$

$x=\frac{z+\bar z}{2}$ $y=\frac{z-\bar z}{2i}$ ... $\bar z$

$y=0$, byt ut x mot z: $f(z)=\sin z+iC$

20

4.1.1 b) $u(x,y) = \cos(2xy) e^{x^2-y^2}$

$\Delta u = 0?$

$$\frac{\partial u}{\partial x} = -\sin(2xy) \cdot 2y e^{x^2-y^2} + \cos 2xy \, e^{x^2-y^2} 2x = \frac{\partial v}{\partial y}$$

$$v = \int \left(-\sin 2xy \cdot 2y \cdot e^{x^2-y^2} + 2x\cos(2xy) e^{x^2-y^2}\right) dy =$$

$$\int \sin(2xy)(e^{x^2-y^2})'_y + \int (\sin 2xy)'_y e^{x^2-y^2} dy = \int \left(\sin(2xy) e^{x^2-y^2}\right)'_y dy =$$

$$= \sin(2xy) e^{x^2-y^2} + \varphi(x)$$

$$\frac{\partial v}{\partial x} = \cos(2xy) \cdot 2y \cdot e^{x^2-y^2} + \sin(2xy) e^{x^2-y^2} \cdot 2x + \varphi'(x) = -\frac{\partial u}{\partial y} =$$

$$= +\left(+\sin(2xy) \cdot 2x e^{x^2-y^2} + \cos(2xy) \cdot e^{x^2-y^2}(+2y)\right) \Rightarrow \varphi'(x) = 0 \Rightarrow \varphi = \text{const} \in \mathbb{R}$$

$$f(z) = \cos 2xy \, e^{x^2-y^2} + i(\sin 2xy \, e^{x^2-y^2} + C)$$

$$f(z) = e^{z^2} + iC$$

e) $u(x,y) = \arctan x$

$$\frac{\partial u}{\partial x} = \frac{1}{1+x^2} \Rightarrow \Delta u \neq 0$$

$$\frac{\partial^2 u}{\partial x^2} = \dots \not\equiv 0$$

$$\frac{\partial^2 u}{\partial y^2} \equiv 0$$

4.1.2) $f: D \to \mathbb{C}$ analytisk fkn. $f \neq 0$ i D

$$\Delta \ln|f| \overset{?}{=} 0 \qquad \boxed{\Delta u = \Delta v = 0} \; C\text{-}R$$

$$|f| = \sqrt{u^2+v^2}$$

$$\varphi(x,y) = 2\ln|f| = \ln|f|^2 = \ln(u^2+v^2)$$

$$\frac{\partial \varphi}{\partial x} = \frac{1}{u^2+v^2}(2uu_x + 2vv_x)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 2\frac{(u^2+v^2)(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}) - 2(uu_x + vv_x)^2}{(u^2+v^2)^2}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \dots \text{ (byt } x \text{ mot } y)$$

$$\Delta\phi = \frac{2}{(u^2+v^2)^2}\Big(\big(u^2+v^2\big)\big(\underbrace{u_x^2+uu_{xx}+v_x^2+vv_{xx}+u_y^2+uu_{yy}}_{u\Delta u}+\underbrace{v_y^2+vv_{yy}}_{v\Delta v}\big)-2\big(u^2u_x^2+2uvu_xv_x+v^2v_x^2$$ (21)

$$+u^2u_y^2+2uvu_yv_y+v^2v_y^2\big)\Big)=$$

$$=\frac{2}{(u^2+v^2)^2}\Big((u^2+v^2)(u_x^2+v_y^2+u_y^2+v_y^2)-2\big((u^2+v^2)(u_x^2+u_y^2)\big)\Big)=0$$

(4.1.12) $\Delta u=0$ (f analytisk $\Rightarrow f^2$ analytisk)

$\Delta(u^2)=0$

$\overset{?}{\Rightarrow} u\equiv \text{const}$

$\frac{\partial}{\partial x}(u^2)=2uu_x$

$\frac{\partial^2}{\partial x^2}(u^2)=2u_x^2+2uu_{xx}$

$0=\Delta(u^2)=2(u_x^2+u_y^2)+2u\Delta u \Rightarrow u_x^2+u_y^2=0$ reella

$\Rightarrow \begin{cases} u_x\equiv 0 \\ u_y\equiv 0\end{cases} \Rightarrow u\equiv\text{const}$ (ty sammanhängande)

(3.3) (4a) Möbiusavb. (3 olika punkter ger entydig cirkel)

$(1,i,-1) \rightarrow (-1,i,1)$

"cirklar" $\rightarrow$ "cirklar"

cirklar, räta linjer

$ad-bc\neq 0$ $\quad f(z)=\frac{az+b}{cz+d}$

$\frac{a+b}{c+d}=-1 \quad a+b+c+d=0$

$\frac{ai+b}{ci+d}=i \quad ai+b+c-di=0$

$\frac{-a+b}{-c+d}=1 \quad -a+b+c-d=0$

$$\begin{cases} a+b+c+d=0 \\ -a+b+c-d=0 \\ ai+b+c-di=0\end{cases}\pm$$

$b+c=0$

$a+d=0$

$(a-d)i+(b+c)=0$

$a-d=0 \rightarrow$

$f(z)=\frac{az+b}{-bz-a}$

(2) $f(z) = \frac{\cancel{b}}{\cancel{-b}z} = -\frac{1}{z}$

(3.34 d) $(\infty, -1, i) \rightsquigarrow (1, 0, 1-i)$

$\overline{\mathbb{C}}$

$f(z) = \frac{az+b}{cz+d} \xrightarrow[z \to \infty]{} \frac{a}{c} = 1 \Leftrightarrow a = c$

$\frac{\cancel{z}(a + \frac{b}{z})}{\cancel{z}(c + \frac{d}{z})}$, $\frac{b}{z} \to 0$, $\frac{d}{z} \to 0$

$f(z) = \frac{az+b}{az+d}$ $\quad \frac{-a+b}{-a+d} = 0 \Leftrightarrow b = a$

(d egentligen $\frac{d}{c}$ pga förk. m. a)

$f(z) = \frac{az+a}{az+d}$

$a \neq 0$ Välj $a = 1$

$f(z) = \frac{z+1}{z+d}$

$\frac{i+1}{i+d} = 1-i$

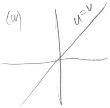
$\cancel{i} + \cancel{1} = \cancel{i} + d + \cancel{1} - di$ $\quad d\underbrace{(1-i)}_{\neq 0} = 0 \Rightarrow d = 0$

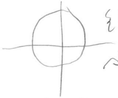
$f(z) = \frac{z+1}{z}$

(6a) $\{|z|=1\}$ på $\operatorname{Re}((1+i)w) = 0$

$w = u+iv$

$\operatorname{Re}((1+i)w) = u - v = 0$ $\quad (w)$





$\{1, -1, i\}$

$\rightsquigarrow \{0, \infty, 1+i\}$

$\frac{az+b}{cz+d}$ $\quad$ "$\frac{-a+b}{\boxed{-c+d}} = \infty$" $\quad \boxed{-c+d} = 0 \Rightarrow \boxed{c = d}$

2,1,15 (Sats) $f: D \to \mathbb{C}$ analytisk (D område)
$f' \equiv 0$ i D

$\Rightarrow f \equiv \text{const}$ i D

Bevis $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv 0 \Rightarrow \begin{cases} \frac{\partial u}{\partial x} \equiv 0 \\ \frac{\partial v}{\partial x} \equiv 0 \end{cases}$ C-R $\Rightarrow \begin{cases} \frac{\partial v}{\partial y} \equiv 0 \\ -\frac{\partial u}{\partial y} \equiv 0 \end{cases}$

$\nabla u \equiv 0$ i D $\Rightarrow u \equiv c_1$ i $D \Rightarrow f \equiv 0$ i D
$\nabla v \equiv 0$ område $v \equiv c_2$

2,1,18 $f(z) = \bar{z}$ ej analytisk i någon pkt i $\mathbb{C}$ (C-R)

Sats $f: D \to \cancel{\mathbb{C}}\ \mathbb{R}$ reellvärd & analytisk

$\Rightarrow f \equiv \text{const}$ i D

Bevis $v \equiv 0$

$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv 0 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \equiv 0 \end{cases} \Rightarrow u \equiv c \Rightarrow f \equiv c$

$|f|$

Sats $f = u + iv$ analytisk i D.
Om $u \equiv \text{const}$ eller $v \equiv \text{const}$, så är $f \equiv \text{const}$

Sats $f: D \to \mathbb{C}$, analytisk
$|f| \equiv \text{const}$
$\Rightarrow f \equiv \text{const}$

Bevis 1) $|f| = 0 \Rightarrow f \equiv 0$ klart
2) $|f| = c \neq 0 \Rightarrow f \neq 0$ i hela D
$|f|^2 = f\bar{f} = c^2 \Rightarrow \bar{f} = \frac{c^2}{f}$ analytisk i D, $f \neq 0$ $\Rightarrow \underbrace{f + \bar{f}}$ analytisk i D
reellvärd $\Rightarrow \underbrace{f + \bar{f}}_{2u} \equiv \text{const}$

$\Rightarrow u \equiv \text{const} \Rightarrow f \equiv \text{const}$

(2,1,25) $f=u+iv$ i D

(24) C-R på polär form

$$\begin{cases} r\dfrac{\partial u}{\partial r}=\dfrac{\partial v}{\partial \theta} \\ r\dfrac{\partial v}{\partial r}=-\dfrac{\partial u}{\partial \theta} \end{cases}$$

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$r(x,y)$

$\theta(x,y)$

$$1=\frac{\partial r}{\partial x}\cos\theta+r(-\sin\theta)\frac{\partial \theta}{\partial x}$$

$$0=\frac{\partial r}{\partial x}\sin\theta+r\cos\theta\frac{\partial \theta}{\partial x}$$

LES (map determinant)

$$\frac{\partial r}{\partial x}=\frac{\begin{vmatrix}1 & -r\sin\theta\\ 0 & r\cos\theta\end{vmatrix}}{\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}}=\frac{r\cos\theta}{r}=\cos\theta=\frac{x}{r}$$

$$\frac{\partial \theta}{\partial x}=\frac{\begin{vmatrix}\cos\theta & 1\\ \sin\theta & 0\end{vmatrix}}{r}=-\frac{1}{r}\sin\theta$$

$$\begin{cases} 0=\dfrac{\partial r}{\partial y}\cos\theta+r(-\sin\theta)\dfrac{\partial \theta}{\partial y} \\ 1=\dfrac{\partial r}{\partial y}\sin\theta+r\cos\theta\dfrac{\partial \theta}{\partial y} \end{cases}$$

$$\frac{\partial r}{\partial y}=\frac{\begin{vmatrix}0 & -r\sin\theta\\ 1 & r\cos\theta\end{vmatrix}}{r}=\frac{r\sin\theta}{r}=\sin\theta$$

$$\frac{\partial \theta}{\partial y}=\frac{\begin{vmatrix}\cos\theta & 0\\ \sin\theta & 1\end{vmatrix}}{r}=\frac{1}{r}\cos\theta$$

$$\begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x}+\dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial x}=\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{1}{r}\sin\theta\dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial y}=\dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial y}+\dfrac{\partial v}{\partial \theta}\dfrac{\partial \theta}{\partial y}=\sin\theta\dfrac{\partial v}{\partial r}+\dfrac{1}{r}\cos\theta\dfrac{\partial v}{\partial \theta} \end{cases}$$

$$\mathbb{C}\begin{cases} \dfrac{\partial u}{\partial y}= \\ \dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial r}\cos\theta+\dfrac{\partial v}{\partial \theta}\dfrac{1}{r}\sin\theta \end{cases}$$

$\rightarrow$

(25) $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$ $\quad -\pi < \operatorname{Arg} z < \pi$

$= \underbrace{\ln r}_{u} + i\underbrace{\theta}_{v}$ $\quad -\pi < \theta < \pi$

$r\frac{\partial u}{\partial r} = r \cdot \frac{1}{r} = 1 = \frac{\partial v}{\partial \theta} = 1$

$r\frac{\partial v}{\partial r} = 0 = -\frac{\partial u}{\partial \theta} = 0$

$\left.\begin{matrix} u, v \in C^1 \\ C\text{-}R \end{matrix}\right\} \Rightarrow \operatorname{Log} z$ analytisk i

Alla entydiga grenar av log är analytiska

$\log_* z = \log_{**} z + 2k\pi i$ (samma snitt)

const

har samma derivata

$\operatorname{Log} z$ $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos\theta \underbrace{\frac{\partial u}{\partial r}}_{\frac{1}{r}} - \frac{1}{r}\sin\theta \underbrace{\frac{\partial u}{\partial \theta}}_{=0} + i\cos\theta \underbrace{\frac{\partial v}{\partial r}}_{=0} - i\frac{1}{r}\sin\theta \underbrace{\frac{\partial v}{\partial \theta}}_{=1}$

$= \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{x - iy}{r^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z}$

4.1.3 $\ln r$ är harmonisk $(z \neq 0)$ $\quad r = |z|$

ty $\ln r = \operatorname{Re} \underbrace{\operatorname{Log} z}_{\text{analytisk}}$, därmed harmonisk i

$\ln r = \operatorname{Re} \log_* z$ harmonisk $\to$

4.1.1c $u(x,y) = \frac{x^2}{x^2+y^2+1}$ $\quad \Delta u \stackrel{?}{=} 0$

Om ja, finn $\underline{v}$

$(\Delta u \neq 0)$ $\quad \frac{\partial u}{\partial x} = \frac{2x(\cancel{x^2}+y^2+1) - \cancel{2x \cdot x^2}}{(x^2+y^2+1)^2} = \frac{2x(y^2+1)}{(x^2+y^2+1)^2} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{x^2 \cdot 2y}{(x^2+y^2+1)^2} = -\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = \frac{2x(\cancel{x^2+y^2+1})}{(x^2+y^2+1)^2} - \frac{2x^3}{(x^2+y^2+1)^2}$ $\quad \Delta u \neq 0$

$v(x,y) = \frac{2x}{\sqrt{\cancel{x^2+1}}} \arctan\frac{y}{\sqrt{x^2+1}} - \dots$

Möbius avbildningar

$f: \bar{\mathbb{C}} \to \bar{\mathbb{C}} \quad \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$
(annars $f \equiv$ const)

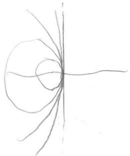
$a, b, c, d \in \mathbb{C}$
analytisk i $\mathbb{C} \setminus \{z = -d/c\}$

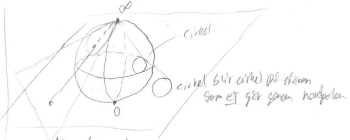
f sammansättning av tre typer av funktioner

- $z+A$ translation
- Bz Omskalning och rotation
- $1/z$?

"cirklar" avbildas på "cirklar"

{ cirklar
{ räta linjer





i $\bar{\mathbb{C}}$: räta linjer är "cirklar" som går genom ∞.

"Cirkel" $\xrightarrow{\text{translation}}$ "cirkel"

" $\xrightarrow{\text{rotation}}$ "

" $\xrightarrow{\text{omskalning}}$ "

Återstår att visa för $\frac{1}{z}$

"cirkel": ekvation:

$Ax^2 + Ay^2 + Bx + Cy + D = 0$

$A = 0$: rät linje
$A \neq 0$: cirkel

$w=\frac{1}{z}$ $\quad z=\frac{1}{w}$ $\quad \frac{\bar{w}}{|w|^2}=\frac{u-iv}{u^2+v^2}$

$w=u+iv$, $z=x+iy$

$x=\frac{u}{u^2+v^2}$, $y=-\frac{v}{u^2+v^2}$

$|z|^2=x^2+y^2=\frac{1}{|w|^2}=\frac{1}{u^2+v^2}$

$A\cdot\frac{1}{u^2+v^2}+B\frac{u}{u^2+v^2}-C\frac{v}{u^2+v^2}+D=0 \;\Big|\,(u^2+v^2)$

$A+Bu-Cv+Du^2+Dv^2=0$ "cirkel" i w-planet.

$\frac{1}{z}$: $\left.\begin{matrix}A=0\\ D\neq 0\end{matrix}\right\}$ rät linje ej genom origo $\longrightarrow$ cirkel genom origo

$A=D=0$ rät linje genom origo $\longrightarrow$ rät linje genom origo

---

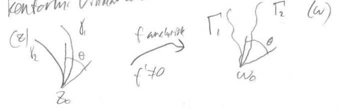
En "cirkel" bestäms entydigt av tre pkter
(rät linje: en pkt i ∞)

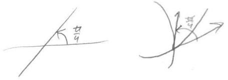
konkret "cirkel" $\xrightarrow{z}$ konkret "cirkel"

välj 3 pkter $\longrightarrow$ välj 3 pkter

---

konformi: vinklar bevaras lokalt till storlek och riktning. $\Delta \longrightarrow D$





28 def $z^{\alpha} = e^{\alpha \log z}$, $z \neq 0$

2006-09-13

$z^{\frac{1}{n}} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + (\theta_0 + 2k\pi)i)} =$

$= \underbrace{e^{\frac{1}{n}(\ln|z| + i\theta_0)}}_{\text{entydigt}} e^{\frac{2k\pi i}{n}} \quad k \in \mathbb{Z}$

n st. olika värden; upprepas

$e^{2k\pi i/n} = ?$

$\underbrace{k = 0, 1, 2, \ldots, n-1}_{n \text{ st.}}$, $k = mn + \underset{\text{rest}}{r}$ $\quad 0 \leq \text{rest} \leq n-1$

$e^{2k\pi i/n} = e^{\frac{2(mn+r)\pi i}{n}} = \underbrace{e^{2m\pi i}}_{1} \cdot e^{\frac{2r\pi i}{n}}$ $\quad 0, 1, \ldots, n-1$; $n, n+1, \ldots$; $2n, \ldots$

$\Rightarrow$ ej fler än $\underline{n}$ olika värden

? $\exists$ n olika

$k_1, k_2 = 0, 1, 2, \ldots, n-1$

$e^{\frac{2k_1\pi i}{n}} = e^{\frac{2k_2\pi i}{n}} \Leftrightarrow e^{2\frac{k_1-k_2}{n}\pi i} = 1 = e^{2\pi i \cdot p}$ (heltal) $\quad$ ty $k_1, k_2 = 0, 1, \ldots, n-1$

$\Leftrightarrow \frac{2(k_1-k_2)\pi i}{n} = 2\pi i p \Leftrightarrow \frac{k_1-k_2}{n} \in \mathbb{Z} \Leftrightarrow k_1 = k_2$

$z^{\frac{1}{n}}$ har exakt $\underline{n}$ st. olika värden.

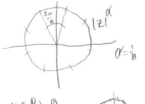
$z^{\frac{1}{n}} = \underbrace{|z|^{\frac{1}{n}}}_{=\sqrt[n]{|z|}} \cdot e^{i(\theta_0 + 2k\pi)/n}$, $k = 0, 2, \ldots, n-1$

$4^{1/2} = \{-2, 2\}$ $\quad \sqrt{} \overset{\text{def}}{=}$ det icke-negativa värde…

$\sqrt{4} = 2$

$\alpha \notin \mathbb{Q} \Rightarrow z^{\alpha}$ har oändligt många olika värden

$\alpha \in \mathbb{R}$



$\alpha \in \mathbb{R} \setminus \mathbb{Q}$