Övning 2006–10–17

Kursens sista övning...

Uppgift 5.1.2′

$$\chi_\sigma(x) = u(x) = \begin{cases} 1, & \text{då } |x| < \sigma \\ 0, & \text{då } |x| > \sigma \end{cases}$$

χ_σ kallas karakteristisk funktion för intervallet $(-\sigma, \sigma)$.

$\Omega \subset \mathbb{R}^n$:

$$\chi_\Omega(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in \Omega \\ 0, & \boldsymbol{x} \notin \Omega \end{cases}$$

$$\int_\Omega f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \chi_\Omega(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}x$$

$$\hat{\chi}_\sigma(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} \chi_\sigma(x)\, \mathrm{d}x = \int_{-\sigma}^{\sigma} \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{d}x =$$

$$[\text{om } \xi \neq 0] = \frac{1}{-\mathrm{i}\xi} \left[\mathrm{e}^{-\mathrm{i}x\xi}\right]_{-\sigma}^{\sigma} = \frac{1}{\mathrm{i}\xi}(-\mathrm{e}^{\mathrm{i}\sigma\xi} + \mathrm{e}^{\mathrm{i}\sigma\xi}) = 2\,\frac{\sin \sigma\xi}{\xi}$$

Om $\xi = 0$:

$$\hat{\chi}_\sigma(0) = \int_{-\sigma}^{\sigma} 1\, \mathrm{d}x = 2\sigma = \lim_{\xi \to 0} 2\,\frac{\sin \sigma\xi}{\xi}$$

Uppgift 5.1.4

$$u(x) = 16\, \mathrm{e}^{-4x^2}$$

$$\hat{u}(\xi) = 16 \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} \cdot \mathrm{e}^{-(2x)^2}\, \mathrm{d}x =$$

$$= 8 \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(2x)\frac{\xi}{2}} \mathrm{e}^{-(2x)^2}\, \mathrm{d}(2x) =$$

$$= 8\sqrt{\pi}\, \mathrm{e}^{-\frac{\left(\frac{\xi}{2}\right)^2}{4}} = 8\sqrt{\pi}\, \mathrm{e}^{-\frac{\xi^2}{16}}$$

Uppgift 5.1.15

$$u(x) = \frac{1}{x^4 + 1}$$

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}x\xi}}{x^4 + 1}\, \mathrm{d}x = [a = -\xi] =$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}ax}}{x^4 + 1}\, \mathrm{d}x$$

(Imaginärdelen är udda, alltså måste vi få ett reellt svar.)

Om $a \geqslant 0$: ⟨fig2⟩

$$\int_{\Gamma_R} f(x)\,\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}ax}}{x^4+1}\,\mathrm{d}x + \int_{C_R} \frac{\mathrm{e}^{\mathrm{i}az}}{z^4+1}\mathrm{d}z$$

$$\mathrm{VL} = 2\pi\mathrm{i} \sum_{\text{öhp}} \operatorname{Res} f$$

$$z^4+1=0 \quad \Rightarrow \quad z = \mathrm{e}^{\mathrm{i}(\pi+2k\pi)/4}, k \in \{0,1,2,3\}$$

$$z_0 = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \mathrm{i}\,\frac{\sqrt{2}}{2}$$

$$z_1 = \mathrm{e}^{\mathrm{i}\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + \mathrm{i}\,\frac{\sqrt{2}}{2}$$

Dessa båda ligger i det övre halvplanet. De båda andra måste då ligga i det nedre.

$$\operatorname{Res}_{z_0} f + \operatorname{Res}_{z_1} f = \left.\frac{\mathrm{e}^{\mathrm{i}az}}{4\,z^3}\right|_{z=z_0} + \left.\frac{\mathrm{e}^{\mathrm{i}az}}{4\,z^3}\right|_{z=z_1} =$$

(Vi har enkla nollställen till nämnaren, täljaren är inte noll.)

$$\left[z_k^4 = -1 \quad \Rightarrow \quad \frac{1}{z_k^3} = -z_k\right]$$

$$= -\frac{1}{4}\mathrm{e}^{\mathrm{i}az_0}\,z_0 - \frac{1}{4}\mathrm{e}^{\mathrm{i}az_1}\,z_1 = -\frac{1}{4}\left(\mathrm{e}^{\mathrm{i}a\frac{\sqrt{2}}{2}} \cdot \mathrm{e}^{-a\frac{\sqrt{2}}{2}}\left(\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}\right) + \mathrm{e}^{-\mathrm{i}a\frac{\sqrt{2}}{2}} \cdot \mathrm{e}^{-a\frac{\sqrt{2}}{2}}\left(-\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}\right)\right) =$$

$$= -\frac{1}{4} \cdot \frac{\sqrt{2}}{2} \cdot \mathrm{e}^{-a\frac{\sqrt{2}}{2}}\left(\mathrm{e}^{\mathrm{i}a\frac{\sqrt{2}}{2}}(1+\mathrm{i}) + \mathrm{e}^{-\mathrm{i}a\frac{\sqrt{2}}{2}}(-1+\mathrm{i})\right) =$$

$$= -\frac{\sqrt{2}}{8} \cdot \mathrm{e}^{-a\frac{\sqrt{2}}{2}} \cdot \left(2\,\mathrm{i} \cdot \sin\frac{a\sqrt{2}}{2} + 2\,\mathrm{i} \cdot \cos\frac{a\sqrt{2}}{2}\right)$$

$$2\,\pi\,\mathrm{i}\,(\operatorname{Res}_{z_0} f + \operatorname{Res}_{z_1} f) = 2\,\pi\,\mathrm{i}\left[-\frac{\sqrt{2}}{8} \cdot \mathrm{e}^{-a\frac{\sqrt{2}}{2}} \cdot \left(2\,\mathrm{i} \cdot \sin\frac{a\sqrt{2}}{2} + 2\,\mathrm{i} \cdot \cos\frac{a\sqrt{2}}{2}\right)\right] =$$

$$= \frac{\pi\sqrt{2}}{2} \cdot \mathrm{e}^{-a\frac{\sqrt{2}}{2}}\left(\sin\frac{a\sqrt{2}}{2} + \cos\frac{a\sqrt{2}}{2}\right)$$

På halvcirkeln:

$$\left|\int_{C_R} \frac{\mathrm{e}^{\mathrm{i}az}}{z^4+1}\,\mathrm{d}z\right| = \begin{bmatrix} z = R\,\mathrm{e}^{\mathrm{i}\theta} \\ 0 \leqslant \theta \leqslant \pi \end{bmatrix} = \left|\int_0^{\pi} \frac{\mathrm{e}^{\mathrm{i}aR\cos\theta} \cdot \mathrm{e}^{-aR\sin\theta}}{R^4 \cdot \mathrm{e}^{4\mathrm{i}\theta}+1} R\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\right| \leqslant$$

$$\leqslant \begin{bmatrix} a \geqslant 0 \\ R > 0 \\ \sin\theta \geqslant 0 \end{bmatrix} \leqslant \pi\,\frac{R \cdot 1}{R^4-1} \longrightarrow 0 \quad \text{då } R \to \infty$$

Nu måste vi i princip betrakta negativa a på den negativa halvcirkeln. $\sin\theta \leqslant 0$, $a \leqslant 0$, så produkten av dem blir positiv.

$$\int_{-\infty}^{\infty} \frac{e^{i a x}}{x^4+1} dx = \int_{-\infty}^{\infty} \frac{\cos a x + i \sin a x}{x^4+1} dx = [\text{udda del tas ut}] =$$

$$= \int_{-\infty}^{\infty} \frac{\cos a x}{x^4+1} dx = [\text{jämn}] \int_{-\infty}^{\infty} \frac{\cos|a|x}{x^4+1} dx$$

I hela uträkningen ovan kan vi sätta $|a|$ istället för a, och $|a| = |\xi|$.

Uppgift 5.3.2 Laplace-uppgift.

$$u(t) = \sinh A t \overset{16}{\supset} \frac{A}{s^2 - A^2}$$

$A \in \mathbb{C}$.

Uppgift 5.3.6

$$u(t) = \begin{cases} 1, & \text{då } 0 < t < \sigma \\ -1, & \text{då } \sigma < t < 2\sigma \\ 0, & \text{då } t > 2\sigma \end{cases}$$

$$\supset \quad \int_0^{\sigma} e^{-st} dt + \int_{\sigma}^{2\sigma} (-1) e^{-st} dt + 0$$

Uttryck u i termer av Heavisides funktion θ. ⟨fig3⟩

$$u(t) = 1 \left(\underbrace{\theta(t) - \theta(t-\sigma)}_{=\chi_{(0,\sigma)}} \right) + (-1) \left(\underbrace{\theta(t-\sigma) - \theta(t-2\sigma)}_{=\chi_{(\sigma,2\sigma)}} \right)$$

$$\mathcal{L}u = 1 \cdot (\mathcal{L}(\theta(t)) - \mathcal{L}(\theta(t-\sigma)) - 1\,(\mathcal{L}(\theta(t-\sigma)) - \mathcal{L}(\theta(t-2\sigma))$$

Uppgift 5.3.14

$$\mathcal{L}u(s) = \frac{1}{(s-1)^2} \quad \underset{n=1, c=1 \in \mathbb{C}}{\overset{12}{\subset}} \quad t^1 e^{1 \cdot t}$$

Uppgift 5.3.13

$$\mathcal{L}u(s) = \frac{s}{(s^2 + A^2)^2}$$

Olika sätt.

a) 10+14+15, faltning av cos och sin.

b) reellt partialbråk. Komplext:

$$\frac{c_1}{s + i A} + \frac{c_2}{(s + i A)^2} + \frac{c_3}{s - iA} + \frac{c_r}{(s - iA)^2}$$

formel 12, $c = \pm i A \in \mathbb{C}$.

c)

$$\mathcal{L}u(s) = \frac{s}{(s^2 + A^2)^2} = -\frac{1}{2A}\left(\frac{A}{s^2 + A^2}\right)' \quad \subset \quad \frac{1}{2\,A}\,t\sin A\,t$$

8+14.

Uppgift 5.5.2 $\{a_j\}_{j=0}^{\infty}$

$$\mathcal{Z}(a_j)(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n} + \cdots$$

$|z| > R$.

$$a_j = \begin{cases} 0, & j \text{ jämnt} \\ (-1)^k \cdot \frac{1}{(2k+1)!} & j = 2k+1, \quad k = 0, 1, 2, \ldots \end{cases}$$

$$\mathcal{Z}(a_j)(z) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2\,k+1)!} \cdot \frac{1}{z^{2k+1}} = \sin\frac{1}{z}$$

Uppgift 5.5.5

$$a_j = \frac{1}{j+1}$$

$$\mathcal{Z}(a_j)(z) = 1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots + \frac{1}{(n+1)\,z^n} + \cdots =$$

$$= z\left(\frac{1}{z} + \frac{1}{2\,z^2} + \cdots + \frac{1}{(n+1)z^{n+1}} + \cdots\right) =$$

$$\overset{?}{=} -z\,\mathrm{Log}\left(1 - \frac{1}{z}\right)$$

$|z| > 1$ gör att $1 - \frac{1}{z}$ inte kommer i närheten av snittet (negativa realaxeln).

$$\lim_{t \to 0} \mathrm{Log}(1-t) \overset{?}{=} 0$$

Ja. $\mathrm{Log}\,1 = 0$.

Uppgift 5.5.8

$$\mathcal{Z}(a_j) = \cos\frac{1}{z}$$

Rekonstruera följden a_j. Laurentutveckla i $\{|z| > R\}$. I det här fallet räcker $|z| > 0$.

$$\cos\frac{1}{z} = 1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \cdots + \frac{(-1)^k}{(2\,k)^1 z^{2k}} + \cdots$$

$$a_j = \begin{cases} 0, & j \text{ udda} \\ \frac{(-1)^k}{(2k)!} & j = 2k, \quad k = 0, 1, 2, \ldots \end{cases}$$