

2006-10-03

räkneövning komplex

2.6.3

Beräkna $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a, b > 0$

$$\text{kv typ } \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$$



eller



$$\deg Q = \deg P + 4$$

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

för singular i punkterna $\pm ai$, $\pm bi$. öhp: ai , bi
 $a+bi$ enkelpoler.

$$\text{Res}_{ai} f = \frac{1}{z^2+b^2} \Big|_{z=ai}$$

$$= \frac{1}{b^2-a^2} = \frac{1}{2ai}$$

$$= \frac{1}{2ai(b^2-a^2)}$$

$$f(z) = \frac{g(z)}{h(z)}$$

Om z_0 är ett
enkelt nollställe
till h ; $g(z_0) \neq 0$.

$$\text{Då gäller } \text{Res}_{z_0} f = \frac{g(z_0)}{h'(z_0)}$$

$$\text{Res}_{bi} f = \frac{1}{z^2+a^2} \Big|_{z=bi}$$

$$= \frac{1}{2bi(a^2-b^2)}$$

$$2\pi i \left(\text{Res}_{ai} f + \text{Res}_{bi} f \right) = 2\pi i \left(\frac{1}{2ai(b^2-a^2)} + \frac{1}{2bi(a^2-b^2)} \right) =$$

$$= \pi \frac{-b+a}{ab(a^2-b^2)} = \frac{\pi}{ab(ab)}$$



$$\int_{\Gamma_R} f(z) dz = \int_{-R}^R \frac{dx}{(x^2+a^2)(x^2+b^2)} + \int_{\Gamma} f(z) dz$$

$$\int_{\Gamma_R} f(z) dz = 2\pi i \sum_{\text{öhp}} \text{Res} f.$$



2.6.3 forts

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$$\left| \int_{\gamma_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} \right| \leq \frac{ML}{2\pi R} \cdot \frac{1}{(R^2-a^2)(R^2-b^2)} \xrightarrow{R \rightarrow \infty} 0$$

$$\frac{\pi}{ab(a+b)} = \int_{\Gamma_R} f(z) dz \xrightarrow{R \rightarrow \infty} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} + 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

då $a \neq b$.

Vad om $a=b$? ① Integralen är en konvergensfunktion av b . $b \rightarrow a$.

② $a=b$, ai är en dubbelpol.

$$\left[\frac{d}{dz} (z-ai)^2 f(z) \right]_{z=ai} = \left(a_2 + a_1(z-ai) + a_0(\dots) \right)$$

$$= \left(\frac{1}{(z+ai)^2} \right)' \Big|_{z=ai} = -2 \frac{1}{(z+ai)^3} \Big|_{z=ai} =$$

$$= +2 \cdot \frac{1}{8a^3i}$$

$$2\pi i \sum \text{Res } f = 2\pi i \frac{1}{4a^3i} = \frac{\pi}{2a^3}$$

(2.6.8)
$$\int_0^{\infty} \frac{\cos \gamma x}{(x^2 + \alpha^2)(x^2 + \beta^2)^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos \gamma x}{(x^2 + \alpha^2)(x^2 + \beta^2)^2} dx$$

Spår $\frac{\pi}{2\alpha\beta(\alpha^2 - \beta^2)} (\alpha e^{-\gamma\beta} - \beta e^{-\gamma\alpha})$

$\alpha = \beta$? Svaret är rätt om $\alpha \neq \beta$, $\alpha, \beta, \gamma > 0$.

(2.6.9)
$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin \theta)^2} = \left[\begin{array}{l} z = e^{i\theta} \\ dz = i e^{i\theta} d\theta \\ d\theta = \frac{1}{iz} dz \end{array} \right] =$$

$$= \int_{|z|=1} \frac{\frac{1}{iz} dz}{(2 - \frac{z - \frac{1}{z}}{2i})^2} = \frac{z - \frac{1}{z}}{2i}$$

$$= \int_{|z|=1} \frac{dz}{iz \cdot \frac{1}{z^2} (2z - \frac{z^2 + 1}{2i})^2} =$$

$$= \frac{1}{i} (-4) \int_{|z|=1} \frac{z dz}{(z^2 - 4iz + 1)^2} = -\frac{4}{i} (2\pi i) \sum_{\substack{\text{innanför} \\ |z|=1}} \text{Res } f =$$

$$\left[\begin{array}{l} z^2 - 4iz + 1 = (z - 2i)^2 + 3 = 0 \\ z - 2i = \pm \sqrt{3}i, \quad z_1 = (2 + \sqrt{3})i \\ z_2 = (2 - \sqrt{3})i \quad \text{innanför } |z|=1 \end{array} \right]$$

$$= -\frac{4}{i} 2\pi i \cdot \text{Res } f_{z_2}$$

 dubbelpol.



2, 6, 9. parts

$$\left((z-z_1)^2 \cdot \frac{z}{\underbrace{(z-z_1)^2(z-z_2)^2}_f} \right)' \bigg|_{z=z_2}$$

$$= \frac{(z-z_1)^2 - 2(z-z_1)z}{(z-z_1)^4} \bigg|_{z=z_2} = \frac{z_2 - z_1 - 2z_2}{(z_2 - z_1)^3} =$$

$$= \frac{-z_1 - z_2}{(z_2 - z_1)^3} = \frac{-4j}{(-2\sqrt{3} \cdot j)^3} = \quad \begin{aligned} z_1 &= (2-\sqrt{3})j \\ z_2 &= (2+\sqrt{3})j \end{aligned}$$

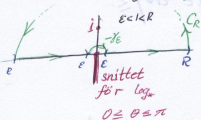
$$= \frac{+4j}{+8 \cdot 3\sqrt{3}(-j)} = -\frac{1}{6\sqrt{3}} \quad \begin{aligned} z_1 + z_2 &= 4j \\ z_2 - z_1 &= -2\sqrt{3}j \end{aligned}$$

$$= \text{Res } f_{z_2}$$

$$\int_0^{2\pi} \frac{d\theta}{(2 - \sin \theta)^2} = +\frac{4}{j} 2\pi j \cdot \frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

2.6.17.

$$\mathcal{D} = \int_0^{\infty} \frac{\ln x}{1+x^2} dx.$$

Generaliserad i 0 och ∞ 

$$\Gamma_{\epsilon,R} = [\epsilon, R] \cup \Gamma_R \cup [-R, -\epsilon] \cup (-\gamma_\epsilon)$$

$$f(z) = \frac{\log_* z}{1+z^2}$$

$$\left\{ \begin{aligned} \int_{\Gamma_{\epsilon,R}} f(z) dz &= \int_{\epsilon}^R \frac{\log_* x}{1+x^2} dx + \int_{\Gamma_R} f(z) dz + \int_{-R}^{-\epsilon} \frac{\log_* z}{1+z^2} dz + \\ &+ \int_{-\gamma_\epsilon} f(z) dz \\ \int_{\Gamma_{\epsilon,R}} f(z) dz &= 2\pi i \operatorname{Res}_i f \end{aligned} \right.$$

$$\operatorname{Res}_i \frac{\log_* i}{2i} = \frac{\ln|i| + i\frac{\pi}{2}}{2i} = \frac{\pi}{4}$$

$$x \in [\epsilon, R]: \log_* x = \ln|x| + i\theta = \ln x$$

$$x \in [-R, -\epsilon]: \log_* z = \ln|z| + i\pi = \ln|x| + \pi i$$

$$\therefore \int_{\epsilon}^R \frac{\log_* x}{1+x^2} dx \rightarrow \int_0^{\infty} \frac{\ln x}{1+x^2} dx$$

$$\begin{aligned} \int_{-R}^{-\epsilon} \frac{\log_* z}{1+z^2} dz &= \int_{-R}^{-\epsilon} \frac{\ln|x| + \pi i}{1+x^2} dx = \int_{-R}^{-\epsilon} \frac{\ln|x|}{1+x^2} dx + \pi i \int_{-R}^{-\epsilon} \frac{dx}{1+x^2} = \\ &= + \int_{\epsilon}^R \frac{\ln t}{1+t^2} dt + \pi i \underbrace{\left[\arctan x \right]_{-R}^{-\epsilon}}_{\pi i \left(0 + \frac{\pi}{2} \right)} \end{aligned}$$

$x=t, dx=dt$



2.6.17. forts

$$\left| \int_{\zeta_R} \frac{\log_x z}{1+z^2} dz \right| = \left| \int_0^\pi \frac{\ln R + i\theta}{1+R^2 e^{2i\theta}} R i e^{i\theta} d\theta \right| \leq$$

$$z = R e^{i\theta}, \theta \in [0, \pi]$$

$$\leq \pi \frac{\ln R + \pi}{R^2 - 1} \cdot R \rightarrow 0 \text{ da } R \rightarrow \infty$$

"interessant, für die gilt
intemot vändligheten"

$$\left| \int_{\zeta_\varepsilon} \frac{\log_x z}{1+z^2} dz \right| \leq \left| \int_\pi^0 \frac{\ln \varepsilon + i\theta}{1+\varepsilon^2 e^{2i\theta}} \cdot \varepsilon i e^{i\theta} d\theta \right| \leq$$

$$\pi \varepsilon_0$$

$$\leq \pi \frac{|\ln \varepsilon| + \pi}{1 - \varepsilon^2} \quad \varepsilon \rightarrow 0$$

$$1 - \varepsilon^2 \rightarrow 1 \text{ da } \varepsilon \rightarrow 0$$

$$|\ln \varepsilon| = -\ln \varepsilon = \ln \frac{1}{\varepsilon}$$

$$\varepsilon \ln \varepsilon \rightarrow 0 \text{ da } \varepsilon \rightarrow 0$$

$$R \rightarrow \infty, \varepsilon \rightarrow 0$$

$$2\pi i \cdot \frac{\pi}{4} = 2\mathcal{J} + 0 + 0 + \pi i \cdot \frac{\pi}{2}$$

$$\mathcal{J} = 0 \quad (\text{Re VL} = \text{Re HL})$$

$$\int_0^\infty \frac{P(x)}{Q(x)} (\ln x)^p dx$$