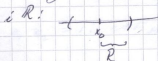


Komplex räkneövning

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Potensserie $\sum_{n=0}^{\infty} a_n z^n$, eller mer allmänt $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

Konvergensradie:



Samma kriterier i $\mathbb{R}$ och $\mathbb{C}$: rot, kvot

2.2.3 $\sum_{k=0}^{\infty} \frac{z^{3k}}{2^k}$ Konvergensradie?

$$\sqrt[n]{\left| \frac{z^{3n}}{2^n} \right|} = \sqrt[n]{\frac{|z|^{3n}}{2^n}} = \frac{|z|^3}{2} \xrightarrow{n \rightarrow \infty} \frac{|z|^3}{2}$$

< 1 för konvergens
 > 1 för divergens
 $= 1$ för ?

$$\frac{|z|^3}{2} < 1 \Leftrightarrow |z| < \sqrt[3]{2}$$

$$\Rightarrow |R = \sqrt[3]{2}|$$

$\sum_{k=0}^{\infty} \left(\frac{z^3}{2} \right)^k$ är en geometrisk serie
 med kvot $\frac{z^3}{2}$

$$\text{konvergent} \Leftrightarrow \left| \frac{z^3}{2} \right| < 1$$

Alternativt:

$$\left| \frac{\text{term}_{n+1}}{\text{term}_n} \right| \rightarrow q \begin{cases} < 1 & \text{konv.} \\ > 1 & \text{div.} \\ = 1 & ? \end{cases}$$

2006-09-19

2.2.17

$$\sum_{n=0}^{\infty} (-1)^n \frac{(z-2\pi i)^n}{n!} = e^{-(z-2\pi i)}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2.2.18

$$\sum_{n=2}^{\infty} n(n-1) z^n$$

Wichtig: andraderivatoren $\sum z^n$

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1) z^n &= \sum_{n=2}^{\infty} n(n-1) z^{n-2} z^2 = \\ &= z^2 \sum_{n=2}^{\infty} n(n-1) z^{n-2} = z^2 \left(\frac{1}{1-z} \right)'' = \dots \end{aligned}$$

2.3.7

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \quad a > b > 0$$

nämmanden $\neq 0$

$$\begin{aligned} 0 \leq \theta \leq 2\pi & \quad z = e^{i\theta} \\ \cos\theta = \frac{z+z^{-1}}{2} & \quad dz = i e^{i\theta} d\theta \\ d\theta = -i \frac{1}{z} dz & \end{aligned}$$

$$I = -i \int_{|z|=1} \frac{\frac{1}{z} dz}{a+b \frac{z+z^{-1}}{2}} = -i \int_{|z|=1} \frac{dz}{az+b \frac{z^2+1}{2}}$$

$$= -2i \int_{|z|=1} \frac{dz}{bz^2+2az+b}$$

$$z_{1,2} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

2006-09-19

$$I = -2i \int_{|z|=1} \frac{dz}{b \left(z - \frac{a - \sqrt{a^2 - b^2}}{b} \right) \left(z - \frac{-a + \sqrt{a^2 - b^2}}{b} \right)}$$

$z_1, z_2 = 1$; ett innanför, ett utanför $|z|=1$
 kvar i nämnaren

$$I = -2i \int_{|z|=1} \frac{1 / \left(b \left[z - \frac{a - \sqrt{a^2 - b^2}}{b} \right] \right)}{z - \frac{-a + \sqrt{a^2 - b^2}}{b}} dz$$

(Cauchy's integralformel)

$$I = \frac{-2i \cdot 2\pi i}{b \left[z - \frac{a - \sqrt{a^2 - b^2}}{b} \right]} \Bigg|_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}}$$

$$I = 4\pi \frac{1}{b \frac{-a + \sqrt{a^2 - b^2} + a + \sqrt{a^2 - b^2}}{b}} =$$

$$= \frac{2\pi}{\sqrt{a^2 - b^2}}$$

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2.3.13

Fresnel-Integrale, $\int_0^\infty \cos(x^2) dx$, $\int_0^\infty \sin(x^2) dx$
(keine elementäre Funktion)



$f(z) = e^{iz^2}$ analytisch

$[0, R]$, $z = x$, $\{|z| = R\}$, $0 \leq \theta \leq \pi/4$, $z = R \cdot e^{i\theta}$

0. Mit $z_0 = z = t z_0$, $t \in [0, 1]$

$z_1 = R \cdot e^{i\pi/4}$, $z = t R \cdot e^{i\pi/4}$, $t \in [0, 1]$

$z = \rho e^{i\pi/4}$, $R \xrightarrow{\rho \rightarrow 0}$

$$\begin{aligned} 0 &= \int_{\Gamma_R} e^{iz^2} dz = \int_0^R e^{ix^2} dx + \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} \cdot R i e^{i\theta} d\theta + \\ &+ \int_R^0 e^{i\rho^2 e^{i\pi/2}} \cdot e^{i\pi/4} d\rho \end{aligned}$$

$$\int_3 = -\frac{i}{2} \int_0^R e^{-\rho^2} d\rho \xrightarrow{R \rightarrow \infty} -\frac{e^{-i\pi/4} \cdot \sqrt{\pi}}{2}$$

Gauß-Integral

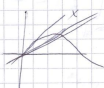
$$|I_2| = \left| \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} e^{i\theta} d\theta \right| =$$

$$= \left| \int_0^{\pi/4} \underbrace{e^{iR^2 \cos 2\theta}}_{|1|=1} \cdot \underbrace{e^{-R^2 \sin 2\theta}}_{|1|=1} \cdot e^{i\theta} d\theta \right| \leq$$

$$\leq R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta = R \int_0^{\pi/4} e^{-R^2 \cdot \frac{2}{\pi} \theta} d\theta$$

$$= R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta + R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta$$

$$< R \cdot 1 \cdot \frac{\pi}{4} < R e^{-R^2 \sin \frac{\pi}{4}} \cdot \frac{\pi}{4}$$



$$\frac{2}{\pi} x \leq \sin x$$

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$$= R \cdot \frac{1}{-\frac{4}{\pi} R^2} \left[e^{-\frac{4R^2}{\pi} \theta} \right]_0^{\pi/4} \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$0 = \int_{\Gamma_R} f(z) dz \xrightarrow{R \rightarrow \infty} \int_0^{\infty} e^{ix^2} dx + 0 - e^{\frac{i\pi}{4}} \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \int_0^{\infty} e^{ix^2} dx = \frac{\sqrt{\pi}}{2} e^{\frac{i\pi}{4}}$$

$$\text{Re: } \int_0^{\infty} \cos(x^2) dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$\text{Im: } \int_0^{\infty} \sin(x^2) dx = \frac{\sqrt{2\pi}}{4}$$

2006-09-19

2.3.14 $f(z_0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z - z_0} dz =$



$\gamma_r: |z - z_0| = r$

$z = z_0 + re^{i\theta}$

$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \cdot re^{i\theta} i d\theta =$

$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$

*Medelvärdessatsen
för harmoniska funktioner.*

2.3.15 $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$

$|f(z)| \leq \frac{1}{2\pi} \max_{\text{cirkeln med radie } r \text{ runt } z} |f| \cdot 2\pi$

$f: D \rightarrow \mathbb{C}$ analytisk. Antag att $|f|$ har lokalt max i z_0 , $z_0 \in D$.

$\Rightarrow |f(z_0)| \geq |f(z)| \quad \forall z: |z - z_0| < \delta$

Tag $r < \delta$: $|f(z_0)| \leq \max_{\text{cirkeln}} |f|$
 $|f(z_0)| \geq \max_{\text{cirkeln}} |f|$

$|f(z_0)| = \max_{\text{cirkeln}} |f| \Rightarrow \text{ej stängt max}$