Komplex räkneövning

2006-09-12

1.5.23 $\quad f(z) = e^z$

$\{ z = x + iy, \quad -\infty < x < \infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \}$

$\xrightarrow{f}$?

rand $\xrightarrow{f}$ rand, utom ?

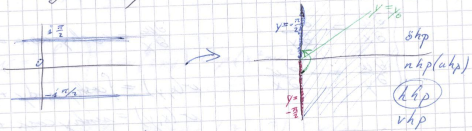
$y = \pm \frac{\pi}{2}, \; y = y_0$

$z = x + iy \Rightarrow e^z = e^x(\cos y + i \sin y)$

$|e^z| = e^x. \qquad 0 < e^x < \infty$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$ är den (en) polära vinkeln för e^z

$\operatorname{Arg} e^z = y$



rand $\xrightarrow{f}$ rand utom 0

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2.1.16 $T(z) = \dfrac{az+b}{cz+d}$ $\quad ad \neq bc$, $a,b,c,d \in \mathbb{C}$

$$T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} =$$

$$= \frac{ad-bc}{(cz+d)^2} \quad \begin{matrix} \neq 0 \text{ för } ad-bc \neq 0 \\ \equiv 0 \text{ för } ad-bc = 0 \end{matrix}$$

2.1.20d $\underset{(lett)}{u(x,y)} = \cosh y \sin x$

Finn det harmoniska konjugatet:

v: $f = u + iv$ analytisk

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\frac{\partial u}{\partial x} = \cosh y \cos x \doteq \frac{\partial v}{\partial y}$$

$$v(x,y) = \int \cosh y \cos x \, dy + \varphi(x)$$

$$v(x,y) = \sinh y \cos x + \varphi(x)$$

$$\frac{\partial v}{\partial x} = -\sinh y \sin x + \varphi'(x) = \underbrace{-\sinh y \sin x}_{-\frac{\partial u}{\partial y}}$$

$$\varphi'(x) \equiv 0$$
$$\varphi(x) \equiv C \in \mathbb{R}$$

$$f(z) = \cosh y \sin x + i(\sinh y \cos x + C)$$

$x = \dfrac{z+\bar{z}}{2}$ $\quad y = \dfrac{z-\bar{z}}{2i}$ $\quad \bar{z}$ bort.

eller: $y=0$, byt ut x mot z:

$$f(z) = \sin z + iC$$

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4.1.1.b $u(x,y) = \cos(2xy)\, e^{x^2-y^2}$

"Är det någon som direkt kan gissa vad $f(z)$ blir?"

$\Delta u = 0$?

$$\frac{\partial u}{\partial x} = -\sin(2xy)\cdot 2y \cdot \exp(x^2-y^2) + \cos(2xy)\exp(\ldots$$

$$\ldots x^2-y^2)\cdot 2x = \frac{\partial v}{\partial y}$$

$$v = \int \left(-\sin(2xy)\cdot 2y \cdot \exp(x^2-y^2) + 2x\cos(2xy)\exp(x^2-y^2)\right) dy$$

$$v = \int \left(\sin(2xy)\, \exp'_y(x^2-y^2)\right) dy + \int \sin'_y(2xy)\exp(x^2+y^2)\, dy$$

$$v = \int \left\{ \sin(2xy)\exp(x^2-y^2) \right\}'_y dy =$$

$$= \sin(2xy)\exp(x^2-y^2) + \varphi(x)$$

$$\frac{\partial v}{\partial x} = \cos(2xy)\cdot 2y \cdot \exp(x^2-y^2) +$$

$$+ \sin(2xy)\cdot\exp(x^2-y^2)\cdot 2x + \varphi'(x)$$

$$\stackrel{!}{=} -\frac{\partial u}{\partial y} = +\left\{ +\sin(2xy)\cdot 2x \exp(x^2-y^2) + \right.$$

$$\left. + \cos(2xy)\exp(x^2-y^2)(+2y) \right\}$$

$$\varphi'(x) \equiv 0 \Rightarrow \varphi \equiv \text{const} \in \mathbb{R}$$

$$f(z) = \cos(2xy)\exp(x^2-y^2) + i\left(\sin 2xy \exp(x^2-y^2) + C\right)$$

$$y = 0 \Rightarrow f(z) = e^{z^2} + iC$$

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4.1.1 e $\quad u(x,y) = \arctan x$

$$\frac{\partial u}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \dots \neq 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta u \neq 0$$

4.1.2 $\quad f: D \to \mathbb{C}$ analytisk, $f \neq 0$ i D

$$\Delta \ln|f| \overset{?}{=} 0$$

$$|f| = \sqrt{u^2+v^2}$$

$\Delta u = \Delta v = 0$ ty f analytisk uppfyller C-R.

Betrakta $\Phi(x,y)$: $\quad 2\ln|f| = \ln|f|^2 =$
$$= \ln(u^2+v^2)$$

$$\frac{\partial \Phi}{\partial x} = \frac{1}{u^2+v^2}\left(2uu_x + 2vv_x\right)$$

$$\frac{\partial^2 \Phi}{\partial x^2} = 2\,\frac{(u^2+v^2)(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}) - 2(uu_x+vv_x)^2}{(u^2+v^2)^2}$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \dots \text{ (byt ut } x \text{ mot } y)$$

$$\Delta\Phi = \frac{2}{(u^2+v^2)^2}\Big\{(u^2+v^2)(u_x^2 + \underbrace{uu_{xx}}_{u\Delta u} + v_x^2 + \underbrace{vv_{xx}}_{v\Delta v} +$$
$$+ u_y^2 + uu_{yy} + v_y^2 + vv_{yy}) - 2(u^2u_x^2 + 2uv\overset{v_y}{u_x}v_x + v^2\underbrace{v_x^2}_{u_y^2} +$$
$$+ u^2u_y^2 + 2uv\underbrace{u_y}_{-v_x}v_y + v^2\underbrace{v_y^2}_{u_x^2})\Big\} =$$

$$= \frac{2}{(u^2+v^2)^2}\Big\{(u^2+v^2)(u_x^2 + \underbrace{v_x^2}_{u_y^2} + u_y^2 + \underbrace{v_y^2}_{u_x^2}) - 2(u^2+v^2)(u_x^2+u_y^2)\Big\} = 0$$

$$\boxed{\Delta \ln|f| = 0}$$

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4.1.12 $\left.\begin{array}{l}\Delta u = 0 \\ \Delta(u^2) = 0\end{array}\right\} \overset{?}{\Rightarrow} u \equiv \text{const}$

(f analytisk $\Rightarrow$ f^2 analytisk)

$$\frac{\partial}{\partial x}(u^2) = 2uu_x$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2u_x^2 + 2uu_{xx}$$

$$\frac{\partial^2}{\partial y^2}(u^2) = 2u_y^2 + 2uu_{yy}$$

$$\Delta(u^2) = 2(u_x^2 + u_y^2) + 2u\underbrace{\Delta u}_{=0}$$

$$u_x^2 + u_y^2 = 0$$

reella $\Rightarrow$ $u_x \equiv u_y \equiv 0$.

område (sammanh.) $u \equiv$ konstant.

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3.3.4a Möbiusavbildning

$(1, i, -1) \mapsto (-1, i, 1)$

"cirklar" $\rightarrow$ "cirkel"

inkl. rät linje

$$f(z) = \frac{az+b}{cz+d}$$

Rent handgripligt

$1:\quad \frac{a+b}{c+d} = -1 \Rightarrow a+b+c+d=0$

$i:\quad \frac{ai+b}{ci+d} = i \Rightarrow ai+b+c-di=0$

$-1:\quad \frac{-a+b}{-c+d} = 1 \Rightarrow -a+b+c-d=0$

$(1)+(i):\quad b+c=0$

$(1)-(i):\quad a+d=0$

$a=-d$

$$f(z) = \frac{az+b}{-bz-a}$$

$(a-d)i+b+c=0 \Rightarrow a=d$

$a=d=0$

$$f(z) = \frac{b}{-bz} = -\frac{1}{z}$$

Vi har en frihetsgrad: kan förlänga med vad som helst.

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$\overline{\mathbb{C}}$

3.3.4.d $\quad (\infty, -1, i) \mapsto (1, 0, 1-i)$

" ∞ är en punkt som är otroligt hemlig att räkna med "

$$f(z) = \frac{az+b}{cz+d} \xrightarrow[z\to\infty]{} \frac{a}{c} \overset{!}{=} 1$$

$$\boxed{a = c}$$

$$f(z) = \frac{az+b}{az+d}$$

$$f(-1) = \frac{-a+b}{-a+d} \overset{!}{=} 0 \Rightarrow \boxed{b=a}$$

$$f(z) = \frac{az+a}{az+d} \quad a \neq 0, \text{ välj } \boxed{a=1:}$$

$$f(z) = \frac{z+1}{z+d}$$

$$f(i) = \frac{i+1}{i+d} = 1-i$$

$$i+1 = i+d+1-di$$

$$0 = d - di \Rightarrow \boxed{d=0}$$

$$f(z) = \frac{z+1}{z}$$

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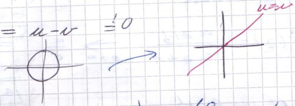
3.3.5a $\{|z|=1\}$ på $Re((1+i)w)=0$

$w=u+iv$; $Re((1+i)w) =$

$= u-v \overset{!}{=} 0$



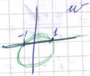
$(1,-1,i) \mapsto (0,\infty,1+i)$

"$\frac{-a+b}{-c+d} = \infty$" $\boxed{c=d}$

⋮

3.3.7d $Re \to Re$

$\{y=x\} \to \{|w+1|=\sqrt{2}\}$



$0 \mapsto -1$

$\infty \mapsto 1$ $a=c=1$

$\frac{z-\alpha}{z+\alpha}$

$Re \to Re \Rightarrow \alpha \in \mathbb{R}$