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Recall:

General linear group: $\mathrm{GL}(n,\mathbb{F})$ where $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$.

Special linear group ($\det g=1$): $\mathrm{SL}(n,\mathbb{F})$. OK for $\mathbb{F}=\mathbb{R},\mathbb{C}$. What if $\mathbb{F}=\mathbb{H}$? We would need a definition of a determinant that defines the order of the factors involved.

Metric preserving groups: We had $G=\mathbb{1}_{n\times n}$:

$$\mathbb{R}:\quad \mathrm{O}(n):\quad g^{\mathrm{T}}G\,g=G$$

$$\mathbb{C}:\quad \mathrm{U}(n):\quad g^{\dagger}G\,g=G$$

$$\mathbb{H}:\quad \mathrm{Sp}(n)\equiv\mathrm{Sp}(n,\mathbb{H}):\quad g^{\dagger}G\,g=G$$

These are all compact groups. That is implied by the metric-preserving condition. There are also non-compact groups:

$$G=\begin{pmatrix}\mathbb{1}_p & \mathbf{0}\\ \mathbf{0} & -\mathbb{1}_q\end{pmatrix},\quad \mathrm{sign}=(p,q)$$

This leads to $\mathrm{O}(p,q)$ and $\mathrm{U}(p,q)$.

We also have $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{C})$: here the preserved metric is anti-symmetric.

$$G=\begin{pmatrix}\mathbf{0} & \mathbb{1}_n\\ -\mathbb{1}_n & \mathbf{0}\end{pmatrix}$$

Example: $g\in\mathrm{SL}(2,\mathbb{R})$.

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad \det g=1\quad\Rightarrow\quad a\,d-b\,c=1.$$

Now consider $g\in\mathrm{Sp}(2,\mathbb{R})$.

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad g^{\mathrm{T}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Rightarrow\begin{pmatrix}0 & a\,d-b\,c\\ b\,c-a\,d & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\Rightarrow\boxed{a\,d-b\,c=1}$$

So $\mathrm{SL}(2,\mathbb{R})\approx\mathrm{Sp}(2,\mathbb{R})$. Both non-compact. Also $\mathrm{SU}(1,1)$ is isomorphic to these.

$$|\alpha|^2-|\beta|^2=1,\quad \alpha,\beta\in\mathbb{C}.$$

Example.

$$\mathrm{SU}(2)/\mathbb{Z}_2\approx\mathrm{SO}(3)$$

(previous lecture). For higher $\mathrm{SO}(n)$ groups we don't have a relation to $\mathrm{SU}(n)$, but need a new name: Spin.

$$\mathrm{Spin}(n)/\mathbb{Z}_2\approx\mathrm{SO}(n)$$

Spin(3) just happens to be SU(2). For low values of n we have:

$\mathrm{Spin}(3)\approx\mathrm{SU}(2)$
$\mathrm{Spin}(4)\approx\mathrm{SU}(2)\times\mathrm{SU}(2)$
$\mathrm{Spin}(5)\approx\mathrm{Sp}(2)$ (compact version)
$\mathrm{Spin}(6)\approx\mathrm{SU}(4)$

And that's the end of that list. No other cases. You have to call Spin(7) by its name.

Discrete groups (subgroups of the Lie groups)

i). $\mathrm{GL}(n,\mathbb{Z})$, meaning that $\det g = \pm 1$.

ii) $\mathrm{SL}(n,\mathbb{Z})$ meaning that $\det g = +1$.

iii) $\mathrm{PSL}(n,\mathbb{Z}) = \mathrm{SL}(n,\mathbb{Z})/\{\mathbb{1}, -\mathbb{1}\}$. This is called the modular group. Important in string theory and all sorts of things.

iv) $\mathrm{O}(n,\mathbb{Z})$, $\det g = \pm 1$. Remember the condition of being compact:

$$\sum_j M_{ij} M_{ij} = 1 \quad (M^{\mathrm{T}} M = \mathbb{1})$$

$\Rightarrow M$ must have at most one non-zero element ± 1 in each row and column.

Example: $\mathrm{O}(3,\mathbb{Z})$:

$$\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} & \pm 1 & \\ & & \pm 1 \\ \pm 1 & & \end{pmatrix}, \quad (\ ), \quad (\ ), \quad (\ ), \quad (\ )$$

Thus $\mathrm{O}(3,\mathbb{Z})$ has $2^3 \times 6 = 48$ elements. $(2^n \times n!)$.

So $\mathrm{O}(3,\mathbb{Z}) \supset S_3$ (or D_3) $\supset A_3$.

Example: $\mathrm{SL}(2,\mathbb{Z})$. This has infinite number of elements.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a\,d - b\,c = 1$$

Statement: for any $(a, b) \in \mathbb{Z}^2$ there is a unique pair $(c, d) \in \mathbb{Z}^2$. This follows from Euclid's algorithm and Bézout's identity. For any (a, b) with gcd = 1 (a and b are coprime: no prime in common), then Euclid's algorithm gives us one (c, d) pair.

Example: $(a, b) = (17, 5)$. Then

$$\begin{aligned} 17 &= 3 \times 5 + 2 \\ 5 &= 2 \times 2 + 1 \text{ (gcd)} \end{aligned}$$

Then $1 = 5 - 2 \times 2 = 5 - 2 \times (17 - 3 \times 5) = -2 \times 17 + 7 \times 5$. Then $(c, d) = (-2, 7)$, satisfying $a\,d + b\,c = 1$.

Other discrete cases $\mathbb{F}_p$. Galois number fields. $\mathrm{GL}(n, \mathbb{F}_p)$, $\det g \neq 0$.

Lie groups are differentiable manifolds. $\mathrm{SU}(2) \approx S^3$. Can we get a metric on these groups?

A canonical metric: with $g \in G$:

$$\omega = g^{-1}\,\mathrm{d}g$$

$g = g(x^i)$.

$\mathrm{SL}(2,\mathbb{R})$, $a\,d - b\,c = 1 \Rightarrow$ 3-dimensional manifold.

$$\omega = \mathrm{d}x^i (\ldots)^i$$

This is called a Maurer-Cartan form.

$$\mathrm{d}s^2 = -\frac{1}{2}\mathrm{Tr}(\omega^2)$$

Use this also on cosets: G/H and H is not invariant ($\mathrm{SO}(8)/\mathrm{SO}(7)\approx S^7$). Metric on S^7:

$$\omega = g\,\mathrm{D}\,g, \quad \mathrm{D}=\mathrm{d}+A, \quad A\in \mathrm{Lie}\,H$$

$$\mathrm{d}s^2 = -\frac{1}{2}\,\mathrm{Tr}(\omega^2)$$

Example: $\mathrm{SU}(2)\approx S^3$. There is a U(1) subgroup. $\mathrm{SU}(2)/\mathrm{U}(1)\approx S^2$. This is the same as saying that $S^3/\mathrm{U}(1)\approx S^2$. That is called Hopf fibration. $S^3 = S^2\times_{\mathrm{tw}}\mathrm{U}(1)$: you need to twist the product. That's the thing you do with the A in the covariant derivative: $\mathrm{D}=\mathrm{d}+A$ where A is a monopole of charge 1.

In fact: $S^7 = S^4\times_{\mathrm{tw}} S^3$ instanton of charge one.

Chapter 4: Lie Algebras

Lie groups are rather trycky objects depending a lot on the choice of coordinates. Issue: how do we determine whether two Lie groups are isomorphic or not? Isomorphic means

i) topologically the same

ii) composition rules must be the same.

This is difficult to use: Reason: (ii) is a very non-linear problem.

Example: $\mathrm{SL}(2,\mathbb{R})$

$$g = \begin{pmatrix} 1+x_1 & x_2 \\ x_3 & \frac{1+x_2x_3}{1+x_1}\end{pmatrix}, \quad \det g = 1 + x_2x_3 - x_2x_3 = 1$$

$$g(x)\,g(y) = g(z), \quad z_i = z_i(x,y)$$

These are very non-linear. Exercise: get them!

Can we simplify the situation? Try to linearize the problem: Let $x_i\to\delta x_i$ small.

Note $g(0,0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$.

Expand g to first order in δx_i:

$$g \simeq \begin{pmatrix} 1+\delta x_1 & \delta x_2 \\ \delta x_3 & 1-\delta x_1\end{pmatrix} + \mathcal{O}(\delta x^2) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \delta x_1\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} + \delta x_2\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + \delta x_3\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} + \mathcal{O}(\delta x^2)$$

So close to the identity element $\mathbb{1} = g(0,0,0)$ we have obtained three matrices:

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$$

called the *generators* of $\mathrm{SL}(2,\mathbb{R})$.

$$T_i = \left.\frac{\partial g}{\partial x_i}\right|_{(0,0,0)}$$

$$\Rightarrow [T_1,T_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0\end{pmatrix} = 2\,T_2$$

$$\begin{cases} [T_1,T_2] = 2\,T_2 \\ [T_1,T_3] = -2T_3 \\ [T_2,T_3] = T_1 \end{cases}$$

Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, called $\mathfrak{sl}(2,\mathbb{R})$.

Example: SU(2). In Euler angles α,β,γ:

$$U(\alpha,\beta,\gamma)=\begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} & -\mathrm{e}^{-\frac{\mathrm{i}}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} \\ \mathrm{e}^{\frac{\mathrm{i}}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} & \mathrm{e}^{\frac{\mathrm{i}}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} \end{pmatrix}$$

The unit element is at $\alpha=\beta=\gamma=0$: $U=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Try to get the generators as

$$\left.\frac{\partial U}{\partial\alpha}\right|_{(0,0,0)}$$

But at $\beta=0$, α and γ appear in the same way.

$$\Rightarrow\frac{\partial U}{\partial\alpha}=\frac{\partial U}{\partial\gamma}\text{ at }(0,0,0).$$

So here we find only two of the three generators. Reason: α,β,γ are bad coordinates at the origin.

Find another set of coordinates:

$$\mathrm{SU}(2):\quad g=\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},\quad |\alpha|^2+|\beta|^2=1$$

$$\Rightarrow\begin{cases} \alpha=x+\mathrm{i}\,y \\ \beta=z+\mathrm{i}w \end{cases}\quad\Rightarrow x^2+y^2+z^2+w^2=1$$

$$\begin{cases} x=\sin\theta_1\sin\theta_2\cos\varphi \\ y=\sin\theta_1\sin\theta_2\sin\varphi \\ z=\sin\theta_1\cos\theta_2 \\ w=\cos\theta_1 \end{cases},\quad\begin{cases} \varphi\colon 0\to 2\pi \\ \theta_1,\theta_2\colon 0\to\pi \end{cases}$$

Now we have to check where the unit matrix sits.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\text{ is at }\theta_1=\theta_2=\frac{\pi}{2},\quad\varphi=0$$

$$\Rightarrow x=1,y=z=w=0.$$

$$\left.\frac{\partial U}{\partial\theta_1}\right|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}=\begin{pmatrix} 0 & \mathrm{i}\frac{\partial w}{\partial\theta_1} \\ \mathrm{i}\,\frac{\partial w}{\partial\theta_1} & 0 \end{pmatrix}=\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}=\mathrm{i}\,\sigma^1$$

$$\left.\frac{\partial U}{\partial\theta_2}\right|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}=-\mathrm{i}\sigma^2$$

$$\left.\frac{\partial U}{\partial\varphi}\right|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}=\mathrm{i}\,\sigma^3$$

$$\left(\frac{\partial z}{\partial\theta_2}\colon z\to -z\quad\Rightarrow -\mathrm{i}\,\sigma^2\to +\mathrm{i}\sigma^2\right)$$

Denote $J_i = \mathrm{i}\,\sigma^i$

$$[J_i, J_k] = -2\varepsilon_{ikj}\, J_k$$

using that $[\sigma^1, \sigma^2] = 2\,\mathrm{i}\,\sigma^3$ etc.

This is the SU(2) Lie algebra!

In Sakurai they talk about $T^a = \frac{1}{2}\sigma^a$. Then $[T^a, T^b] = \mathrm{i}\,\mathrm{e}^{abc}T^c$. These are Hermitian. The J_i are anti-Hermitian.

Example: SO(3). Rotations around the z-axis (or in the xy plane):

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \phi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\phi^2)$$

This works for any rotation in any plane, in any dimension. The xy generator

$$T^3 = T^{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or

$$(T^{12})_{kl} = -2\,\delta^{12}_{kl}$$

$$\delta^{ij}_{kl} \equiv \frac{1}{2}\left(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l\right)$$

In general

$$(T^{ij})_{kl} = -2\,\delta^{ij}_{kl}$$

In three dimensions we can use the notation with rotation axis rather than rotation plane:

$$(T^i)_{kl} \sim \varepsilon^{ikl}$$

The Lie algebra:

$$[T^{ij}, T^{kl}] = 4\delta^{[i}_{[k}T^{j]}{}_{l]}$$

Exercise: Check!

A better definition for SO(N) is

$$(M^{ij})_{kl} = \delta^{ij}_{kl}$$

$$\Rightarrow [M^{ij}, M^{kl}] = -8\,\delta^{[i}_{[k}M^{j]}{}_{l]}$$

In physics (General Relativity) we use covariant derivatives with spin connections: $\omega_\mu \in$ Lie algebra of SO(N). Acting on Lorentz vectors: $\omega_\mu{}^k{}_j V^j = \omega_{\mu ij}(M^{ij})^k{}_l V^l$. Now it is conveniant to have defined $(M^{ij})^k{}_l$ without any strange factors.

The SO(3) Lie algebra for the T^a's:

$$[T^a, T^b] = -\varepsilon^{abc}T^c$$

This is identical to SU(2) Lie algebra. But the T^a are now real 3×3 and antisymmetric, i.e. anti-hermitian like the J_i in $\mathfrak{su}(2)$ above related by $-\frac{1}{2}$ to $J_i = +\frac{\mathrm{i}}{2}\sigma_i$.