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Last time we started on continuous groups.

Chapter 3: Continuous groups

Recall: We looked at rotations in three dimensions:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

In index notation we write this as

$$x^i \to x'^{\,i} = R^i{}_j \, x^j = x^j (R^{\mathrm{T}})_j{}^i$$

The scalar product is invariant under rotations:

$$x'^i \, x'_i = x'^{\,i} \, x'^{\,i} = (x^j R^{\mathrm{T}\,i}_j)\big(x^k R^{\mathrm{T}\,i}_k\big) = x^i R^{\mathrm{T}\,i}_j R^i_k \, x^k = \boldsymbol{x}^{\mathrm{T}} \, R^{\mathrm{T}} \, R \, \boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$$

$$\Rightarrow R^{\mathrm{T}} R = \mathbb{1}.$$

There are two issues here.

1) Rotations leave the scalar product invariant, or equivalently, the metric δ_{ij} invariant:

$$\delta_{ij} \xrightarrow{R} \delta'_{ij} = R^k_i R_j{}^l \delta_{kl} \overset{\text{inv.}}{=\!=} \delta_{ij} \quad \Leftrightarrow \quad R \, \mathbb{1} \, R^{\mathrm{T}} = \mathbb{1}$$

2) Covariant and contravariant indices. Don't introduce the scalar product δ_{ij}. Then we have $x^i \to x'^{\,i} = x^j R_j{}^i$. This implies that $x_i \to x'_i = (R^{-1})_i{}^j x_j$. (We can't use the transpose anymore, because we haven't introduced the metric δ_{ij}.)

$$x'^{\,i} x'_i = x^j R_j{}^i (R^{-1})_i{}^k x_k = \boldsymbol{x}^{\mathrm{T}} R \, R^{-1} \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{x}$$

Here $R_i{}^k \in \mathrm{GL}(3, \mathbb{R})$, the general linear group.

Note. In General Relativity an invariant is the *exterior derivative*:

$$\mathrm{d} = \mathrm{d}x^\mu \, \partial_\mu$$

d is invariant under general coordinate transformations.

$$x^\mu \to x'^{\,\mu} = x'^{\,\mu}(x).$$

Everything follows from this relation. If you differentiate you get

$$\mathrm{d}x'^{\,\mu} = \mathrm{d}x^\nu \left(\frac{\partial x'^{\,\mu}}{\partial x^\nu} \right).$$

For the derivative we consider the inverse transformation

$$x'^{\,\mu} \to x^\mu = x^\mu(x').$$

$$\partial'_\mu = \left(\frac{\partial x^\nu}{\partial x'^{\,\mu}} \right) \partial_\nu$$

Then

$$\mathrm{d}x'^{\mu}\,\partial'_{\mu} = \mathrm{d}x^{\nu}\underbrace{\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)}_{=\delta_{\nu}^{\rho}}\partial_{\rho} = \mathrm{d}x^{\mu}\,\partial_{\mu}$$

This is the first step into differential geometry. The group in General Relativity is $\mathrm{Diff}(M)$ — the diffeomorphisms on the manifold M. This is different form the general linear group:

$$\mathrm{GL}(n,\mathbb{R}) = \begin{pmatrix} a & b & c \\ . & . & . \\ . & . & . \end{pmatrix} = \text{const (independent of } x^{\mu}).$$

Example: The Standard Model. Here we have a so called *gauge group* $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. Take, for instance, the left-handed quarks, which sit in a doublet:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L: \quad (*)_{\alpha,i}, \qquad \begin{matrix} \alpha = 1\colon & u \\ \alpha = 2\colon & d \end{matrix}, \qquad i\colon \text{color index (three values)}.$$

The matrices of the groups act on the corresponding index of the wave-function. Exactly why the groups are $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ is not known. All particles of nature can be fit into this scheme. The index α is an $\mathrm{SU}(2)$ index, the color index i is an $\mathrm{SU}(3)$ index.

Note: "Representations" are matrices for group elements $g \in G$. "Modules" = vector spaces on which these representations act. *But* physicists often call both of these "representations".

Now we will leave this, which was a number of examples, and discuss more generally what kind of groups one can have.

We will now discuss more general classes of groups defined by matrices. These are called *matrix groups* or classical Lie groups.

Start by considering completely general $n \times n$ real matrices.

$$A = \begin{pmatrix} a_1{}^1 & a_1{}^2 & \cdots & a_1{}^n \\ \vdots & & & \\ a_n{}^1 & a_n{}^2 & \cdots & a_n{}^n \end{pmatrix}, \qquad a_i{}^j \in \mathbb{R}$$

Then if A and B are such matrices, so is C defined by $C = A\,B$. Now $A, B, C \in \mathrm{GL}(n,\mathbb{R})$ is a group defined by $\det A \neq 0$, which is required for the inverse A^{-1} to exist.

Recall:

$$\det A = \frac{1}{n!}\,\varepsilon^{i_1 \ldots i_n}\varepsilon_{j_1 \ldots j_n}\,a_{i_1}{}^{j_1} \ldots a_{i_n}{}^{j_n} = \varepsilon^{i_1 \ldots i_n}\,a_{i_1}{}^1 a_{i_2}{}^2 \ldots a_{i_n}{}^n.$$

Then

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \text{Exercise!} \\ \text{Fill in this matrix} \end{pmatrix}$$

Also: We can discuss A with $a_i{}^j \in \mathbb{C}$ and $\det A \neq 0$. Then $A \in \mathrm{GL}(n,\mathbb{C})$.

What else can we do? We can take $a_i{}^j \in \mathbb{H}$ (quaternions). The octonions are non-associative, so they are out.

Complex numbers $z \in \mathbb{C}$ can be written $z = a + \mathrm{i}\,b$ with $\mathrm{i}^2 = -1$.

The quaternions $q \in \mathbb{H}$ can be written $q = q_0 + \mathrm{i}\, q_1 + \mathrm{j}\, q_2 + \mathrm{k} q_3$ where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$ and $\mathrm{i}\,\mathrm{j} = -\mathrm{j}\,\mathrm{i}$. These i, j, k are basically Pauli matrices.

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the only "normed division algebras" $|z_1 z_2| = |z_1|\,|z_2|$.

With the octonions $\mathbb{O}$ we have one real and seven imaginary units e_i. $e_i^2 = -1$. $e_1 e_2 = e_4 = -e_2 e_1$. $\langle$fig$\rangle$. $(e_1 e_2) e_3 \neq e_1 (e_2 e_3)$

Finally: $\mathrm{GL}(n, \mathbb{F}_{p^n})$ with a number field where p is a prime, and p^n is the n'th power of p.

Exercise: Look up definitions of "ring" and "field" and compare to the concept of a group.

Exercise: Construct the addition table and multiplication table of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\} = \mathbb{Z}(\mathrm{mod}\, 5)$. Compare to $\mathbb{Z}_6$. Something is happening here: the prime is important.

A few things that have played a very important role in the development of theoretical physics are monopoles and instantons. The instantons are sort of monopoles that use the (4-dimensional) spacetime as a whole. Here the octonions are important. If you looked at the way 't Hooft did it, for example.

Let us now play a little with these matrices.

Special Linear Group: $\det A = +1$.

$$\mathrm{SL}(n, \mathbb{R})$$

if $AB = C$ with $A, B \in \mathrm{SL} \Rightarrow C \in \mathrm{SL}$. $\det \in \mathbb{R}$. $\det = 1$ is one condition.

$$\mathrm{SL}(n, \mathbb{C})$$

if $AB = C$ with $A, B \in \mathrm{SL} \Rightarrow C \in \mathrm{SL}$. $\det \in \mathbb{C}$. $\det = 1$ is two conditions.

$$\mathrm{SL}(n, \mathbb{H})$$

if $AB = C$ with $A, B \in \mathrm{SL} \Rightarrow C \in \mathrm{SL}$. $\det \in \mathbb{H}$. $\det = 1$ is four conditions. Exercise: Is this possible? Does the definition make sense?

Dimension of the groups are:

$$\begin{array}{ll} \mathbb{R}: & n^2 - 1 \\ \mathbb{C}: & 2\,n^2 - 2 \\ (\mathbb{H}: & 4\,n^2 - 4) \end{array}$$

Other subgroups can be obtained as follows.

Example

$$(\ \mathrm{GL}\ ) \rightarrow \begin{pmatrix} \cdot & \cdot \\ 0 & \cdot \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} a\,\alpha & a\,\beta + b\,\gamma \\ 0 & c\,\gamma \end{pmatrix}$$

This is evidently a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Example

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\,\alpha & a\,\beta + b \\ 0 & 1 \end{pmatrix}$$

The Poincaré group can be written rather like this.

Two special cases are

$$\mathrm{Sol}(n)$$

This means “solvable”. (When we do Lie algebras it will be clearer what these words mean.)

$$S = \begin{pmatrix} s_1 & \times & \times & \times \\ & s_2 & \times & \times \\ & & \ddots & \times \\ \mathbf{0} & & & s_n \end{pmatrix}$$

$$\mathrm{Nil}(n)$$

This means “nilpotent”.

$$N = \begin{pmatrix} 1 & \times & \times & \times \\ & 1 & \times & \times \\ & & \ddots & \times \\ \mathbf{0} & & & 1 \end{pmatrix}$$

Example: Nil(3) is the Heisenberg group.

Additional requirements of preserving some metric give:

Orthogonal groups:

$$R \in O(n, \mathbb{R})$$

They satisfy $R^T G R = G$ where G is the trivial metric $G = \mathbb{1}$.

This is a *compact group*. (Exercise: Check this.) That is an enormously important property. When we talk about representations later: all can be made unitary.

With

$$G = \begin{pmatrix} \mathbb{1}_p & \mathbf{0} \\ \mathbf{0} & -\mathbb{1}_q \end{pmatrix} \quad \Rightarrow \quad O(p, q; \mathbb{R})$$

which is non-compact. $x^2 + y^2 - z^2 = 1$.

This you can repeat for complex numbers.

Unitary groups:

$$U \in \mathrm{GL}(n, \mathbb{C}) \text{ such that } U^\dagger U = \mathbb{1}$$

or with $G = \mathbb{1}$:

$$U^\dagger G U = G$$

Then the non-compact versions satisfy $U^\dagger G U = G$ with

$$G = \begin{pmatrix} \mathbb{1}_p & \mathbf{0} \\ \mathbf{0} & -\mathbb{1}_q \end{pmatrix}$$

$$U \in \mathrm{U}(p, q).$$

Symplectic groups: $M \in \mathrm{Sp}(2\,n, \mathbb{R}), \mathrm{Sp}(2\,n, \mathbb{C})$

These preserve an anti-symmetric G:

$$G=\begin{pmatrix} \mathbf{0} & \mathbb{1}_n \\ -\mathbb{1}_n & \mathbf{0} \end{pmatrix}$$

Note: Light-cone variables in physics: If you take $-x_0^2+x_1^2+x_2^2+x_3^2$ (Minkowski space). Make the first sector of the metric anti-diagonal: Exercise.

$$\rightarrow\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To combine different requirements we need the theorem:

If $H_1\subset G$ and $H_2\subset G$ then $H_{12}:=H_1\cap H_2$ is also a subgroup of G.

Example: $\mathrm{SL}(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R})$ and $O(p,q;\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R})$. This implies that $\mathrm{SO}(p,q;\mathbb{R})\subset\mathrm{GL}(n,\mathbb{R})$ where $\mathrm{SO}(p,q)=\mathrm{SL}(n)\cap O(p,q)$.

also

$$\mathrm{SU}(p,q)=\mathrm{U}(p,q)\cap\mathrm{SL}(p+q,\mathbb{C})$$

Example: $\mathrm{SU}(1,1)=?$ Recall

$$\mathrm{SU}(2)=\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad\Rightarrow \det=|a|^2+|b|^2=1 \quad\Rightarrow \mathrm{SU}(2)\sim S^3$$

$$\begin{cases} a=x_1+\mathrm{i}\,x_2 \\ b=x_3+\mathrm{i}\,x_4 \end{cases} \Rightarrow \quad \sum_{i=1}^{4} x_i^2=1 \quad\Rightarrow S^3$$

$$\mathrm{SU}(1,1)=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad\Rightarrow \det=|a|^2-|b|^2=1$$

$$\Rightarrow x_1^2+x_2^2-x_3^2-x_4^2=1$$

This is called *hyperbolic* geometry.

Example: $\mathrm{SL}(2,\mathbb{R})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{GL}(2,\mathbb{R})$$

$$\det=1=a\,d-b\,c$$

Exercise: Compare $\mathrm{SU}(1,1)$ and $\mathrm{SL}(2,\mathbb{R})$.

Recall the concepts from finite groups: OK also for Lie groups.

Def: Subgroup.

Def: Proper subgroups

Def: Invariant subgroups

Def: Simple group.

Def: Semi-simple: If G has no abelian (proper) invariant subgroups it is semi-simple.

Note: We will later classify all simple groups (via their Lie algebra) and find some examples that are not in the classical matrix groups discussed above. These are called exceptional. G_2, F_4, E_6, E_7, E_8. They are related to octonions.

Subgroups and cosets

We know that $\mathrm{SU}(2) \approx S^3$. But what is SU(3) for instance? It is tricky. It is easy to get the algebraic equation, but to see what the manifold is, that's tricky. $\mathrm{SO}(3) \approx \mathrm{RP}^3$. $\mathrm{SO}(5) = ?$

Cosets: If H is invariant, then G/H is a group. But if H is not invariant, then G/H is not a group. Just some manifold (that's actually important).

EXAMPLE:

$$\mathrm{SO}(14)/\mathrm{SO}(13) = S^{13}$$

$$\mathrm{SO}(3)/\mathrm{SO}(2) = S^2$$

The isometry group of a manifold divided by the isotropy group is the manifold itself.

Center:

Recall the relation between SU(2) and SO(3) in e.g. Quantum Mechanics.

In general; Consider $\mathrm{SU}(N)$ and all elements in it that commute with all elements in $\mathrm{SU}(N)$.

$$g = \ldots \mathbb{1}, \quad \text{satisfy } \det g = 1, \quad \Rightarrow \quad g = \alpha\, \mathbb{1} \text{ where } \alpha^N = 1 \quad \Rightarrow \quad \alpha = \mathrm{e}^{2\pi \mathrm{i} \frac{n}{N}}, n \in \{0, 1, \ldots, N-1\}$$

Thus $\alpha \in \mathbb{Z}_N$.

Now

$$\mathrm{SU}(N)/\mathbb{Z}_N = \text{group}$$

since $\mathbb{Z}_N$ is an invariant subgroup.

EXERCISE: Check this!

EXAMPLE:

SU(2) and $\mathrm{SU}(2)/\mathbb{Z}_2 \approx \mathrm{SO}(3) \approx \mathrm{RP}^3$.

That it is SO here is just a fluke, in the $\mathrm{SU}(N)$ case this is not normally so.