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Problem 4.1.

> “Write out the covariant Laplacian $\mathrm{D}_\mu \mathrm{D}^\mu \equiv g^{\mu\nu}\,\mathrm{D}_\mu \mathrm{D}_\nu$ acting on a scalar for a two-dimensional flat space using polar coordinates.”

Write out the covariant Laplacian $\mathrm{D}_\mu \mathrm{D}^\mu$ acting on a scalar field ϕ for a two-dimensional flat space.

The covariant derivative

$$\mathrm{D}_\mu \phi = \partial_\mu \phi \quad (\phi \text{ scalar})$$

$$\mathrm{D}_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho \quad (A_\nu \text{ covariant vector field})$$

For flat two-dimensional space in polar coordinates:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \quad \Rightarrow \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

Thus, writing out $\mathrm{D}_\mu \mathrm{D}^\mu$:

$$\mathrm{D}_\mu \mathrm{D}^\mu \phi = g^{\mu\nu} \mathrm{D}_\mu \mathrm{D}_\nu \phi = g^{\mu\nu} \mathrm{D}_\mu (\mathrm{D}_\nu \phi) = g^{\mu\nu} \mathrm{D}_\mu \underbrace{(\partial_\nu \phi)}_{\text{vector}} = g^{\mu\nu} \big(\partial_\mu \partial_\nu \phi - \Gamma^\rho_{\mu\nu} \partial_\rho \phi\big) =$$

$$= g^{rr}\Big(\partial_r^2 \phi - \cancel{\Gamma^\rho_{rr} \partial_\rho \phi}\Big) + g^{\theta\theta}\big(\partial_\theta^2 \phi - \Gamma^\rho_{\theta\theta} \partial_\rho \phi\big) = g^{rr}\big(\partial_r^2 \phi\big) + g^{\theta\theta}\big(\partial_\theta^2 \phi - \Gamma^r_{\theta\theta} \partial_r \phi\big) =$$

$$= \partial_r^2 \phi + \frac{1}{r^2}\big(\partial_\theta^2 \phi + r\,\partial_r \phi\big) = \left[\partial_r^2 + \frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\partial_\theta^2\right]\phi$$

Problem 4.3

> “Calculate the curvature tensor, the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R on the surface of a two-sphere of radius a embedded in a Euclidean three-dimensional space.”

Calculate $R^\lambda{}_{\mu\nu\kappa}$, $R_{\mu\nu}$, R for S^2_a. From before

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1 & 0 \\ 0 & \csc^2\theta \end{pmatrix}$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

The Riemann tensor:

$$R^\lambda{}_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\kappa} - \Gamma^\rho_{\mu\kappa} \Gamma^\lambda_{\rho\nu}$$

$$R^\theta{}_{\varphi\theta\varphi} = \cancel{\partial_\varphi \Gamma^\theta_{\varphi\theta}} - \partial_\theta \Gamma^\theta_{\varphi\varphi} + \underbrace{\Gamma^\rho_{\varphi\theta}\,\Gamma^\theta_{\rho\varphi}}_{=\Gamma^\varphi_{\varphi\theta}\Gamma^\theta_{\varphi\varphi}} - \cancel{\Gamma^\rho_{\varphi\varphi}\Gamma^\theta_{\rho\theta}} =$$

$$= -\big(\sin^2\theta - \cos^2\theta\big) + \frac{\cos\theta}{\sin\theta}(-\sin\theta\cos\theta) = -\sin^2\theta$$

$$R_{\theta\varphi\theta\varphi} = g_{\theta\theta} R^\theta{}_{\varphi\theta\varphi} = -a^2 \sin^2\theta = R_{\varphi\theta\varphi\theta}$$

$$R_{\varphi\theta\theta\varphi} = R_{\theta\varphi\varphi\theta} = -R_{\theta\varphi\theta\varphi} = a^2 \sin^2\theta$$

$$R_{\theta\theta**} = R_{\varphi\varphi**} = R_{**\theta\theta} = R_{**\varphi\varphi} = 0$$

The Ricci tensor:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\sigma\mu\rho\nu} = g^{\rho\sigma} R_{\rho\nu\sigma\mu} = R_{\nu\mu} \quad \text{(symmetric)}$$

$$R_{\theta\theta} = g^{\rho\sigma} R_{\sigma\theta\rho\theta} = g^{\varphi\varphi} R_{\varphi\theta\varphi\theta} = \frac{1}{a^2 \sin^2\theta}\left(-a^2 \sin^2\theta\right) = -1$$

$$R_{\varphi\varphi} = g^{\theta\theta} R_{\theta\varphi\theta\varphi} = \frac{1}{a^2}\left(-a^2 \sin^2\theta\right) = -\sin^2\theta$$

$$R_{\theta\varphi} = R_{\varphi\theta} = g^{\rho\sigma} R_{\sigma\varphi\rho\theta} = g^{\varphi\varphi} R_{\varphi\varphi\varphi\theta} = 0$$

$$R_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & -\sin^2\theta \end{pmatrix}$$

The curvature scalar

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} = g^{\theta\theta} R_{\theta\theta} + g^{\varphi\varphi} R_{\varphi\varphi} = \frac{1}{a^2}(-1) + \frac{1}{a^2 \sin^2\theta}(-\sin^2\theta) = -\frac{2}{a^2}$$

Note: in two dimensions $R^{\lambda}{}_{\mu\nu\kappa}$ has only one independent component, and there is a simple relation between $R^{\lambda}{}_{\mu\nu\kappa}$ and R (Weinberg, page 143)

$$R_{\lambda\mu\nu\rho} = \frac{1}{2} R \left(g_{\lambda\nu} g_{\mu\rho} - g_{\lambda\rho} g_{\mu\nu}\right)$$

For S^2:

$$R_{\theta\varphi\theta\varphi} = \frac{1}{2}\left(-\frac{2}{a^2}\right)\left(g_{\theta\theta} g_{\varphi\varphi} - \cancel{g_{\theta\varphi} g_{\theta\varphi}}\right) = -\frac{1}{a^2}\left(a^2 \cdot a^2 \sin^2\theta\right) = -a^2 \sin^2\theta$$

This worked out OK.

Problem 3.4

> "Perform a parallel transport of a contravariant vector along a 'latitude' on a two-sphere."

A vector $A^{\mu}(\tau)$ defined over some path $x^{\mu}(\tau)$ is said to be parallel transported along $x^{\mu}(\tau)$ if the vector satisfies

$$\frac{\mathrm{D}A^{\mu}}{\mathrm{D}\tau} = \frac{\mathrm{d}A^{\mu}(\tau)}{\mathrm{d}\tau} + \Gamma^{\mu}_{\rho\sigma} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} A^{\sigma} = 0 \tag{1}$$

For S^2 we have

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^{\varphi}_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$$

Study the components of (1):

$\mu = \theta$:

$$\frac{\mathrm{d}A^{\theta}}{\mathrm{d}\tau} + \Gamma^{\theta}_{\rho\sigma} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} A^{\sigma} = \frac{\mathrm{d}A^{\theta}}{\mathrm{d}\tau} + \Gamma^{\theta}_{\varphi\varphi} \frac{\mathrm{d}x^{\varphi}}{\mathrm{d}\tau} A^{\varphi} = \frac{\mathrm{d}A^{\theta}}{\mathrm{d}\tau} - \sin\theta\cos\theta\, \dot{\varphi}\, A^{\varphi} \tag{2}$$

$\mu = \varphi$:

$$\frac{\mathrm{d}A^{\varphi}}{\mathrm{d}\tau} + \Gamma^{\varphi}_{\rho\sigma} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} A^{\sigma} = \frac{\mathrm{d}A^{\varphi}}{\mathrm{d}\tau} + \Gamma^{\varphi}_{\theta\varphi}\left(\dot{\varphi}\, A^{\theta} + \dot{\theta}\, A^{\varphi}\right) = \frac{\mathrm{d}A^{\varphi}}{\mathrm{d}\tau} + \frac{\cos\theta}{\sin\theta}\left(\dot{\varphi}\, A^{\theta} + \dot{\theta}\, A^{\varphi}\right) = 0 \tag{3}$$

Let $\tilde{A}^{\varphi} = \sin\theta\, A^{\varphi} \Rightarrow$

$$\frac{\mathrm{d}\tilde{A}^{\varphi}}{\mathrm{d}\tau} = \cos\theta\, \dot{\theta}\, A^{\varphi} + \sin\theta \frac{\mathrm{d}A^{\varphi}}{\mathrm{d}\tau} \tag{4}$$

(3) and (4) $\Rightarrow$

$$\begin{cases} \frac{\mathrm{d}A^\theta}{\mathrm{d}\tau} = \cos\theta\,\dot{\varphi}\,\tilde{A}^\varphi \\ \frac{\mathrm{d}\tilde{A}^\varphi}{\mathrm{d}\tau} = -\cos\theta\,\dot{\varphi}\,A^\theta \end{cases}$$

A latitude is a circle on S^2 with $\theta = \theta_0 =$ constant. The natural way to parametrise a latitude is to use $\tau = \varphi$ as the parameter

$$\Rightarrow \begin{cases} \frac{\mathrm{d}A^\theta}{\mathrm{d}\varphi} = \cos\theta_0 \tilde{A}^\varphi \\ \frac{\mathrm{d}\tilde{A}^\varphi}{\mathrm{d}\varphi} = -\cos\theta_0 A^\theta \end{cases}$$

$$\Rightarrow \frac{\mathrm{d}^2 A^\theta}{\mathrm{d}\varphi^2} + \cos^2\theta_0\, A^\theta = 0$$

This equation can be solved to obtain $A^\theta(\varphi)$ and $\tilde{A}(\varphi)$.

$$\Rightarrow \begin{cases} A^\theta(\varphi) = C_1 \cos(\cos\theta_0\,\varphi) + C_2 \sin(\cos\theta_0\,\varphi) \\ \tilde{A}^\varphi(\varphi) = \frac{1}{\cos\theta_0} \frac{\mathrm{d}A^0}{\mathrm{d}\varphi} = -C_1 \sin(\cos\theta_0\,\varphi) + C_2 \cos(\cos\theta_0\,\varphi) \end{cases}$$

To see how parallel transport affects a vector

$$A^\mu = (A^\theta, A^\varphi) = \boldsymbol{e}_\theta, \quad \text{i.e. } A^\theta = 1, A^\varphi = 0 \text{ at } \varphi = 0$$

$$A^\theta(\varphi = 0) = C_1 = 1$$

$$\tilde{A}^\varphi(\varphi = 0) = C_2 = 0$$

$$\Rightarrow \begin{cases} A^\theta(\varphi) = \cos(\cos\theta_0\,\varphi) \\ A^\varphi(\varphi) = -\sin(\cos\theta_0 \varphi)/\sin\theta_0 \end{cases}$$

After one orbit along the latitude $\theta = \theta_0$

$$A^\theta(\varphi = 2\pi) = \cos(\cos\theta_0 \cdot 2\pi) \neq 1 \text{ in general}$$

$$A^\varphi(\varphi = 2\pi) = -\frac{1}{\sin\theta_0} \sin(\cos\theta_0 \cdot 2\pi) \neq 0 \text{ in general}$$

$\Rightarrow$ The vector A^μ has acquired a nonzero φ-component $\Rightarrow$ Curvature!

Note: $\theta_0 = \pi/2 \Rightarrow A^\theta(2\pi) = 1, A^\varphi(2\pi) = 0$. The reason is that $\theta_0 = \pi/2$ is a geodesic.

A few words on action.

The action of a free particle in curved space-time: Parametrise the trajectory with λ. The action is $A = \int \mathrm{d}\tau = \int \mathrm{d}\lambda \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\mathrm{d}\lambda$. For a massive particle we can use τ as our parameter.

$$A = \int \underbrace{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}_{L=1}\mathrm{d}\tau$$

$$\Rightarrow L = 1, \quad \frac{\mathrm{d}L}{\mathrm{d}\tau} = 0$$

Let $L' = L^2 = -\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. Euler-Lagrange:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L'}{\partial \dot{x}^\mu}\right) - \frac{\partial L'}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}t}\left(2\,L\,\frac{\partial L}{\partial \dot{x}^\mu}\right) - 2\,L\,\frac{\partial L}{\partial x^\mu} = \underbrace{2\,\frac{\mathrm{d}L}{\mathrm{d}t}\,\frac{\partial L}{\partial \dot{x}^\mu}}_{=0} + 2\,L\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu}\right] = 0$$

$\Rightarrow L' = -\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ can be used as the Lagrangian.

Problem 3.3

> "Determine all time-like and light-like geodesics for the two-dimensional metric $\mathrm{d}\tau^2 = t^4 \mathrm{d}t^2 - t^2 \mathrm{d}^2$."

Time-like: (massive particles). We use τ as the parameter.

$$L = -\, g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = t^4\,\dot{t}^2 - t^2 \dot{x}^2 = 1$$

Euler-Lagrange for x:

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = -\,2\,t^2\,\dot{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \Rightarrow \dot{x} = \frac{C_1}{t^2} \text{ for a constant } C_1$$

Two cases: $C_1 = 0$: $\dot{x} = 0 \Rightarrow x = C_2$. From the Lagrangian

$$1 = t^4 \dot{t}^2 - 0$$

$$\Rightarrow \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \pm\frac{1}{t^2} \Rightarrow t^2 \mathrm{d}t = \pm\,\mathrm{d}\tau$$

$$\Rightarrow \frac{1}{3}t^3 = \pm\,\tau + \frac{1}{3}C_3 \quad \Rightarrow t = (\pm\,3\tau + C_3)^{1/3}$$

$C_1 \neq 0$:

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}t}{\mathrm{d}x} = \dot{x}\,\frac{\mathrm{d}t}{\mathrm{d}x}$$

From the Lagrangian:

$$1 = t^4 \dot{t}^2 - t^2 \dot{x}^2 = \left(t^4\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 - t^2\right)\dot{x}^2 = C_1^2\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 - C_1^2\,\frac{1}{t^2}$$

$$\Rightarrow \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \left(\frac{1}{C_1^2} + \frac{1}{t^2}\right)^{-1}$$

$$\int \mathrm{d}x = \pm \int \mathrm{d}t\left(\frac{1}{C_1^2} + \frac{1}{t^2}\right)^{-1/2} = \pm \int \mathrm{d}t\,\frac{t}{\left(1 + \frac{t^2}{C_1^2}\right)^{1/2}}$$

$$\Rightarrow x(t) = \pm\,C_1\left(1 + \frac{t^2}{C_1^2}\right)^{1/2} + C_4$$

Light-like: (massless particle)

$$\mathrm{d}\tau^2 = t^4 \mathrm{d}t - t^2 \mathrm{d}x^2 = 0$$

$$\Rightarrow \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = t^2 \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \pm\,t$$

$$\Rightarrow x(t) = \pm\,\frac{1}{2}\,t^2 + C_5$$