2009–02–11

Lecturer: Niclas Wyllard

Thermodynamics and the evolution of the universe

Thermodynamics can be used to study the evolution of the universe in the hot Big Bang model. In particular, one can follow the number densities of the various particles / radiation. This can be used to obtain predictions for certain aspects of the universe we observe today: abundance of the light elements (H, He, ...), remnant photons and neutrinos, etc.

When and why can thermodynamics be used to describe the evolution of the universe? Maintaining thermodynamic equilibrium requires frequent interactions between the quanta of the system. One can use thermodynamical quantities like temperature, pressure, volume, etc to describe the system (universe).

If Γ is the interaction rate per particle then $\Gamma \gg H$ has to hold to maintain equilibrium. In other words, the interaction rate has to be larger than the expansion rate. However, typically Γ decreases faster with temperature than H does, which means that at certain times some particles will leave thermodynamical equilibrium. The number density of such particles will then be "frozen" at some value, and is subsequently only affected by the expansion. "Freeze out" is an important mechanism that explains much of the particle content of the universe today.

Equilibrium thermodynamics

In the dilute, weakly-interacting gas approximation the distribution function for a particle species i is given by (with Boltzmann's constant $k_{\mathrm{B}} = 1$):

$$f_i(\boldsymbol{p}) = \frac{1}{\mathrm{e}^{(E_i - \mu_i)/T} \pm 1}, \qquad \begin{array}{ll} -: & \text{Bose-Einstein (for bosons)} \\ +: & \text{Fermi-Dirac (for fermions)} \end{array}$$

Here: $E_i = \sqrt{\boldsymbol{p}^2 + m^2}$ and μ_i is the chemical potential. The chemical potential potential [*sic*] can often be neglected in the early universe. The number density of species i is:

$$n_i = \frac{g_i}{(2\pi)^3} \int f_i(\boldsymbol{p})\, \mathrm{d}^3 p$$

where g_i is the number of polarisations (helicity states).

$$\begin{array}{ll} g = 2 & \text{electron} \\ g = 2 & \text{photon} \\ g = 3 & W^{\pm}, Z \end{array}$$

Similarly the energy density is

$$\rho_i = \frac{g_i}{(2\pi)^3} \int E_i\, f_i(\boldsymbol{p})\, \mathrm{d}^3 p$$

and the pressure is

$$p_i = \frac{g_i}{(2\pi)^3} \int \frac{|\boldsymbol{p}|^2}{3\, E_i}\, f_i(\boldsymbol{p})\, \mathrm{d}^3 p$$

In certain limiting cases the above integrals can be evaluated exactly. In the non-relativistic case $T \ll m$ one finds (the same result for bosons and fermions):

$$\begin{cases} n_{\mathrm{NR}} = g_i \left(\frac{m\, T}{2\pi}\right)^{3/2} \mathrm{e}^{-m/T} \\ \rho_{\mathrm{NR}} = m \cdot n_{\mathrm{NR}} \\ p_{\mathrm{NR}} = T \cdot n_{\mathrm{NR}} \ll \rho_{\mathrm{NR}} \end{cases}$$

In the relativistic limit $T \gg m$ one finds

$$n_{\mathrm{R}} = \begin{cases} \frac{\zeta(3)}{\pi^2}\, g_i\, T^3 & \text{(bosons)} \\ \frac{3}{4}\cdot\frac{\zeta(3)}{\pi^2}\, g_i T^3 & \text{(fermions)} \end{cases}$$

$$\rho_{\mathrm{R}} = \begin{cases} \frac{\pi^2}{30}\, g_i\, T^4 & \text{(bosons)} \\ \frac{7}{8}\cdot\frac{\pi^2}{30}\, g_i T^4 & \text{(fermions)} \end{cases}$$

$$p_{\mathrm{R}} = \rho_{\mathrm{R}}/3.$$

Conservation of entropy

Recall˙

$$T\,\mathrm{d}S = \mathrm{d}(\rho(T)\,V) + p(T)\,\mathrm{d}V$$

$$\mathrm{d}S(V,T) = \frac{\partial S}{\partial V}\,\mathrm{d}V + \frac{\partial S}{\partial T}\,\mathrm{d}T$$

$$\Rightarrow \frac{\partial S}{\partial V} = \frac{1}{T}\,(\rho + p); \qquad \frac{\partial S}{\partial T} = \frac{V}{T}\,\frac{\mathrm{d}\rho}{\mathrm{d}T}$$

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V} \quad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}T} = \frac{1}{T}\,(\rho + p)$$

The first law becomes

$$\mathrm{d}S = \frac{1}{T}\,\mathrm{d}((p+\rho)\,V) - \frac{V}{T^2}(\rho + p)\,\mathrm{d}T = \mathrm{d}\left(\frac{1}{T}(\rho + p)\,V\right)$$

Thus

$$S = \frac{V}{T}(\rho + p)$$

Now, from Friedmann's equations we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\rho a^3\big) = -\,p\,\frac{\mathrm{d}}{\mathrm{d}t}\,a^3$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho V) = -\,p\,\frac{\mathrm{d}V}{\mathrm{d}t} \quad \Rightarrow \quad \frac{\mathrm{d}S}{\mathrm{d}t} = 0\right)$$

$$\Leftrightarrow a^3\,\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(a^3(p+\rho)\big)$$

Using

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{1}{T}(\rho + p) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{T}\right) = -\,\frac{1}{T^2}\,\frac{\mathrm{d}T}{\mathrm{d}t}$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(\frac{a^3}{T}(\rho + p)\right)}_{=S\ \text{with}\ V=a^3} = 0.$$

Since the energy density (and number density) of non-relativistic particles are negligible compared to those of relativistic particles it is a good good [*sic*] approximation to sum over only the relativistic particles that are in thermal equilibrium at a given T.

In other words

$$\rho \approx \frac{\pi^2}{30} g_{\rm eff} T^4$$

where

$$g_{\rm eff} = \sum_{\substack{i=\text{relativistic}\\ \text{bosons}}} g_i + \frac{7}{8} \sum_{\substack{i=\text{relativistic}\\ \text{fermions}}} g_i.$$

Example: At a temperature where all the particles of the standard model are relativistic (e.g. $T \sim 1\,\text{TeV}$).

$$g_{\rm eff} = 28 + \frac{7}{8} \times 90 = 106.75$$

$$28 = \underbrace{8 \times 2}_{\substack{\text{SU(3)}\\ \text{gluons}}} + \underbrace{3 \times 3}_{\substack{\text{SU(2)}\\ W^{\pm}, Z}} + \underbrace{2}_{\substack{\text{U(1)}\\ \gamma}} + \underbrace{1}_{\text{Higgs}}$$

$$90 = \underbrace{3 \times 2 \times 2}_{e,\mu,\tau} + \underbrace{3 \times 2}_{\nu} + \underbrace{6 \times 3 \times 2 \times 2}_{\text{quarks}}$$

Particles whose interaction rate becomes smaller than the expansion rate may still be relativistic and have a thermal spectrum. This can be taken into account by introducing a specific temperature T_i for each particle. $g_{\rm eff}$ is the

$$g_{\rm eff} = \sum_{\substack{i\\ \text{relativistic}\\ \text{bosons}}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\substack{i\\ \text{relativistic}\\ \text{fermions}}} g_i \left(\frac{T_i}{T}\right)^4$$

Recall that the Friedmann equations imply that

$$H^2 = \frac{8\pi G}{3} \rho_{\rm R} = \frac{8\pi G}{3} \frac{\pi^2}{30} g_{\rm eff} T^4 \approx 2.76 \frac{g_{\rm eff} T^4}{m_{\rm Pl}^2}$$

$$\Rightarrow H = 1.66 \sqrt{g_{\rm eff}} \frac{T^2}{m_{\rm Pl}}$$

For a radiation dominated universe $a \sim \sqrt{t} \Rightarrow H \equiv \frac{\dot{a}}{a} = \frac{1}{2t}$ which combined with the above result gives

$$t = 0.3 \frac{m_{\rm Pl}}{\sqrt{g_{\rm eff}}} \frac{1}{T^2} \sim \left(\frac{1\,\text{MeV}}{T}\right)^2 \text{seconds.}$$

Entropy density $s \equiv S/V = (\rho + p)/T$. Using the same methods as before (and $p = \rho/3$) one finds

$$s = \frac{2\pi^2}{45} g_{\rm eff}^s T^3$$

$$g_{\rm eff}^s = \sum_{\substack{i\\ \text{relativistic}\\ \text{bosons}}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\substack{i\\ \text{relativistic}\\ \text{fermions}}} g_i \left(\frac{T_i}{T}\right)^3.$$