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Spin 1:

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}), \quad \mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + \mathrm{i}\,g\,[\mathbb{A}_\mu, \mathbb{A}_\nu]$$

Gauge invariance:

$$\delta \mathbb{A}_\mu = \partial_\mu \boldsymbol{\alpha} - \mathrm{i}\,g\,[\mathbb{A}_\mu, \boldsymbol{\alpha}]$$

Spin 0:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \phi\,\partial^\mu \phi - V(\phi)$$

The Higgs mechanism. A way to generate mass. The spin 1 particles cannot have a mass term in its Lagrangian, due to gauge invariance — but we see massive vector particles.

Let's study the theory with a complex scalar field ϕ and gauge field A_μ in more detail. For definiteness let us consider the case $V(\phi^*\phi) = \lambda\Big(\phi^*\phi - \frac{v^2}{2}\Big)^2$.

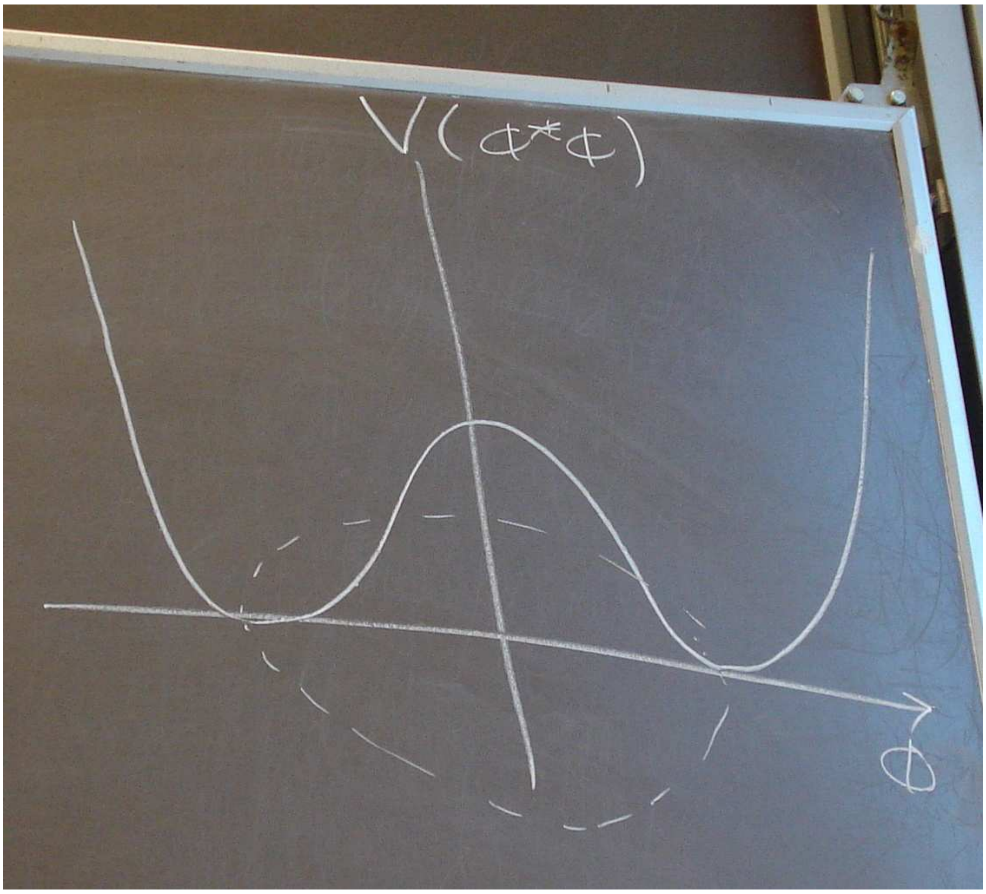

Figure 1. The Mexican hat potential

The potential energy is minimised when $\phi^*\phi = v^2/2$, i.e.

$$\phi = \frac{v}{\sqrt{2}}\,\mathrm{e}^{\mathrm{i}\theta}, \quad \theta \text{ arbitrary.}$$

Let us write

$$\phi = \left(\frac{v}{\sqrt{2}} + \rho(x)\right) \mathrm{e}^{\mathrm{i}e\zeta(x)}$$

Inserting this into the Lagrangian and defining $N_\mu = A_\mu + \partial_\mu \zeta$, we find:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho\, \partial^\mu \rho - \lambda\left(\sqrt{2}\,\rho\, v + \rho^2\right)^2 - \frac{1}{4} F^{\mu\nu}(N)\, F_{\mu\nu}(N) + \frac{e^2}{2}\left(\frac{V}{\sqrt{2}} + \rho\right)^2 N_\mu N^\mu$$

where $F_{\mu\nu}(N) = \partial_\mu N_\nu - \partial_\nu N_\mu$. This Lagrangian is gauge invariant since $A_\mu \to A_\mu + \partial_\mu \alpha$ and $\zeta \to \zeta - \alpha$ implies that $N_\mu \to N_\mu$. But something has happened: the vector field N_μ has acquired a mass term! One says that A_μ has eaten ζ and becomes massive. (ζ appears nowhere in $\mathcal{L}$). Before A_μ had two degrees of freedom and ϕ two. Now N_μ has three degrees of freedom and ρ has one. This way of generating a mass term for spin 1 fields is called the Higgs mechanism. Note that N_μ is invariant under gauge transformations, and does *not* transform as $N_\mu \to N_\mu + \partial_\mu \alpha$ like A_μ does.

One says that the gauge invariance is "spontaneously broken" (although it is not really broken, just hidden). The mass of ρ (the Higgs particle) is $m^2 = 4\,\lambda\, v^2$.

Spin 1/2 fields (fermions)

A spin 1/2 field with mass m is described by a Lagrangian of the form

$$\mathcal{L} = \mathrm{i}\,\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

Here γ^μ are four 4×4 dimensional matrices obeying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \mathbf{1}_{4\times 4} \cdot 2\eta^{\mu\nu}$$

These matrices γ^μ are called Dirac matrices.

ψ is a four-component object.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

ψ is not really a vector. It is a spinor. Just think of it as a four-component object. The entries are *anti*commuting: $\psi_1 \psi_2 = -\,\psi_2 \psi_1$, etc. Finally $\bar{\psi} = \psi^\dagger \gamma^0$. The equation of motion that follows from the above Lagrangian is the Dirac equation:

$$\mathrm{i}\,\gamma^\mu \partial_\mu \psi - m\psi = 0$$

A spin 1/2 field can be coupled to a gauge field via:

$$\mathcal{L} = \mathrm{i}\,\bar{\psi}\gamma^\mu \mathrm{D}_\mu \psi - m\,\bar{\psi}\psi$$

D_μ: "$\partial_\mu + \mathrm{i}\, e\, A_\mu$"

This is gauge invariant under

$$\begin{cases} \psi \to \mathrm{e}^{-\mathrm{i}e\alpha}\psi \\ A_\mu \to A_\mu + \partial_\mu \alpha \end{cases}$$

Comment: ψ describes both a (charged) spin 1/2 particle (e.g. the electron) and its antiparticle (of opposite charge, e.g. the positron).

Quarks and leptons

In addition to the particles mediating the forces and the Higgs particle, the standard model also contains a host of spin 1/2 particles. They are of two types: quarks and leptons.

In the first category, the quarks: u (up), d (down), c (charm), s (strange), t (top), b (bottom).

Then, the leptons: e (electron), ν_e (electron neutrino), μ (muon), ν_μ (muon neutrino), τ (tau), ν_τ (tau neutrino).

The difference is in the interactions. Leptons do not feel the strong force. Quarks feel both the strong and the weak forces.

The quarks come in three kinds (colours): red, green, blue.

$$\begin{matrix}\text{SU(2)}\\ \text{doublet}\end{matrix}\left\{\underbrace{\begin{pmatrix} u_L^r & u_L^g & u_L^b \\ d_L^r & d_L^g & d_L^b \end{pmatrix}}_{\text{SU(3) triplet}}\right. \quad \begin{pmatrix} c_L^r & c_L^g & c_L^b \\ s_L^r & s_L^g & s_L^b \end{pmatrix} \quad \begin{pmatrix} t_L^r & t_L^g & t_L^b \\ b_L^r & b_L^g & b_L^b \end{pmatrix}$$

L above means left-handed.

$$\begin{matrix}\text{SU(2)}\\ \text{doublet}\end{matrix}\left\{\begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}\right. \quad \begin{pmatrix} \nu_{\mu,L} \\ e_L \end{pmatrix} \quad \begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix}$$

The corresponding antiparticles also exist. The right-handed particles are SU(2) singlets, and the neutrino may not have a right-handed partner.

Electric charges:

$$\begin{array}{rl} u, c, t\text{:} & +2/3 \\ d, s, b\text{:} & -1/3 \\ e, \mu, \tau\text{:} & -1 \end{array}$$

Confinement, mesons, baryons

Free quarks have not been observed. Quarks only exists as bound states (at least at low energies). This is known as confinement. The allowed bound states have no colour.

- Mesons are formed of a quark and an anti-quark (e.g. red + anti-red = white).

$$\pi^+ = u\,\bar{d} \quad \text{(up–anti-down)}$$

- Baryons are formed of three quarks (red + green + blue = white)

$$p = u\,u\,d \quad \text{(proton)}, \quad n = u\,d\,d \quad \text{(neutron)}$$

More on Higgs. The Higgs mechanism is believed to be realised in nature in the SU(2) $\times$ $U(1)$ electroweak theory.

$$\mathcal{L} = \frac{1}{2}(\mathbb{D}_\mu\Phi)^\dagger(\mathbb{D}^\mu\Phi) - V(\Phi^\dagger\Phi) - \frac{1}{4}\sum_{a=1}^{3} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \text{where}$$

$$\mathbb{D}_\mu = \mathbf{1}_{2\times 2}\,\partial_\mu + \mathrm{i}\,g_2\,\mathbb{W}_\mu - \mathrm{i}\,g_1\,B_\mu\,\mathbf{1}_{2\times 2}$$

is a 2×2 matrix and Φ is a 2-component vector.

$$\mathbb{W}_\mu = \frac{1}{2}\begin{pmatrix} w_{3,\mu} & w_{1,\mu} - \mathrm{i}\,w_{2,\mu} \\ w_{1,\mu} + \mathrm{i}w_{2,\mu} & -\,w_{3,\mu} \end{pmatrix}$$

with

$$V(\Phi^\dagger\Phi) = \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2$$

One can show:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho + \lambda\,v^2\,\rho^2 + \lambda\,v\,\rho^3 + \frac{\lambda}{4}\rho^4 + \frac{1}{8}\big(g_1 B_\mu - g_2 W_\mu^3\big)\big(g_1 B^\mu - g_2 W^{3\mu}\big)(V+\rho)^2 +$$

$$+\frac{1}{8}g_2^2\big(W_\mu' - \mathrm{i}W_\mu^2\big)\big(W^{1\mu} + \mathrm{i}W^{2\mu}\big)(V+\rho)^2\text{:}\quad \Phi \text{ dependent terms.}$$

Not diagonal in B_μ, W_μ^3. Define:

$$Z_\mu = \cos\theta_w W_\mu^3 - \sin\theta_w\,B_\mu,\quad A_\mu = \sin\theta_w W_\mu^3 + \cos\theta_w\,B_m\quad (\text{this } A_\mu \text{ is the photon})$$

where $\cos\theta_w = g_2/\sqrt{g_1^2+g_2^2}$, then get

$$-\frac{1}{4}F_{\mu\nu}(Z)\,F^{\mu\nu}(Z) - \frac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) + \frac{1}{2}\,m_Z^2 Z_\mu Z^\mu$$

$$m_Z = \frac{1}{2}\,v\,\left(g_1^2+g_2^2\right)$$

$$m_{W\pm} = \frac{1}{4}v^2 g_2^2$$

Experimentally

$$m_W \simeq 80.4\,\text{GeV}$$

$$m_Z \simeq 91.2\ \text{GeV}$$

$$\sin^2\theta_W \simeq 0.23$$

(Weinberg angle)