2008–12–10

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Calculate the total cross section σ_{tot}:

$$\sigma_{\text{tot}} = \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

The differential cross section depends on the angle and the incoming energy of the particle. Let us consider the case when a point particle is scattered of a hard sphere.

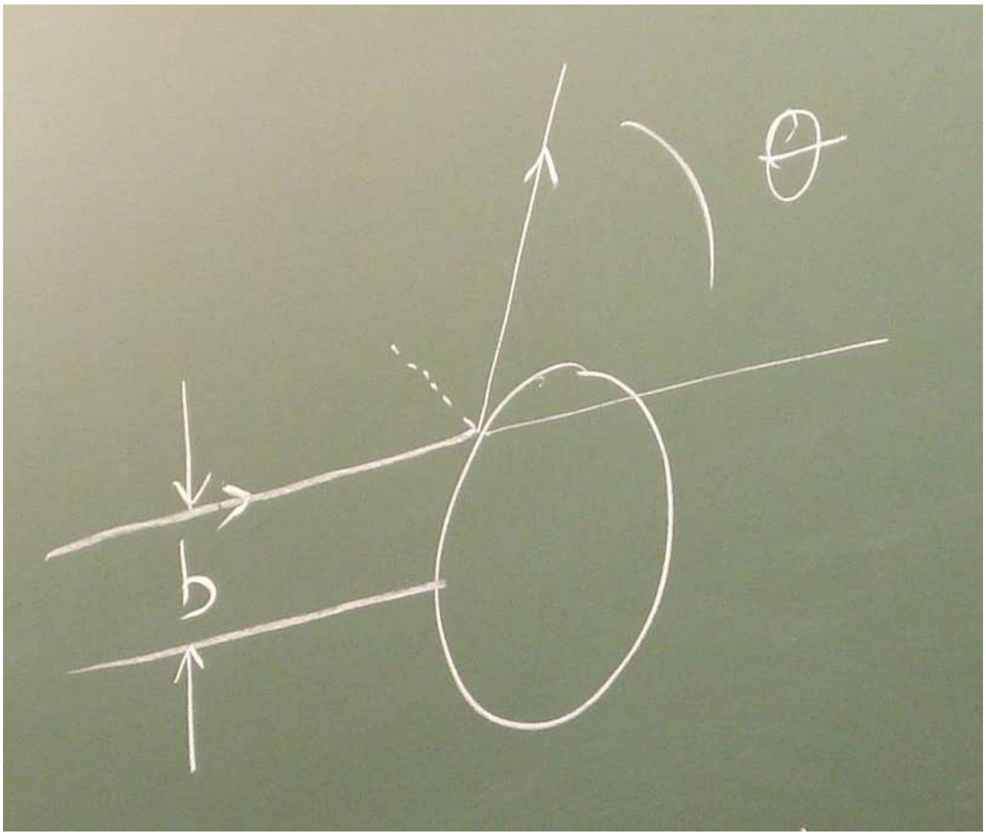

Figure 1. b is the impact parameter, a is the radius of the sphere.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

The classical result becomes:

$$\sigma_{\text{tot}} = \pi a^2$$

Quantum mechanically:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f_{\boldsymbol{k}}(\theta, \varphi)|^2$$

$$f_{\boldsymbol{k}}(\theta, \varphi) = \frac{1}{k}\sum_{l=1}^{\infty} (2l+1)\, \mathrm{e}^{\mathrm{i}\delta_l} \sin\delta_l\, P_l(\cos\theta) \tag{1}$$

$k\,a \gg 1$. $\sigma_{\text{tot}} = 2\,\pi\,a^2$.

Eikonal approximation. It comes from optics, where the eikonal is just the action S. You can view it as a "classical" approximation for δ_l.

$$l\,\hbar = b\,\hbar\,k, \quad l = b\,k$$

(Solution of radial equation $u_l(r) \sim A\,l\sin\left(r\,r - \frac{l\pi}{2} + \delta_l\right)$ for $r \to \infty$.)

WKB scattering theorem Consider a one-dimensional potential.

$$\psi(x) \sim \frac{A}{\sqrt{p(x)}}\sin\left(\frac{1}{\hbar}\int_{x_0}^{x} \mathrm{d}x'\,p(x') + \frac{\pi}{4}\right)$$

Exact asymptotic form of scattering wavefunction.

$$\psi(x) \sim \sin(k\,x + \delta)$$

The phases should be equal.

$$k\,x + \delta = \frac{1}{\hbar}\int_{x_0}^{x} \mathrm{d}x'\,p(x') + \frac{\pi}{4} \text{ when } x \text{ is large}$$

WKB phase shift:

$$\delta = \lim_{x \to \infty} \frac{1}{\hbar}\int_{x_0}^{x} \mathrm{d}x'\,p(x') - k\,x + \frac{\pi}{4}$$

This formula has a geometric interpretation in phase space.

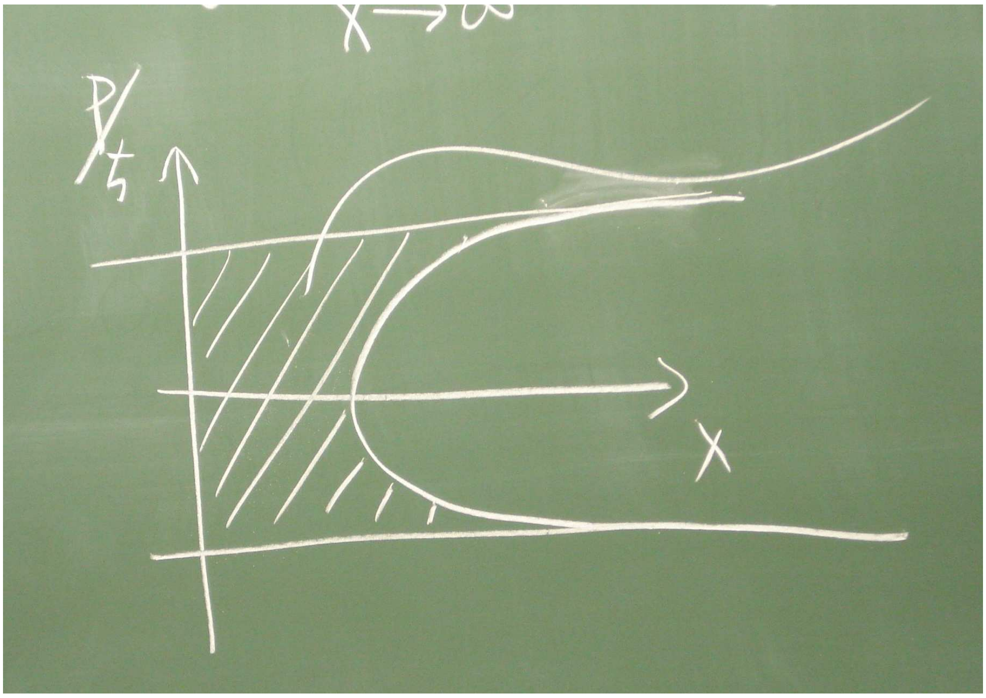

Figure 2. Phase space. The area is $\delta - \frac{\pi}{4}$.

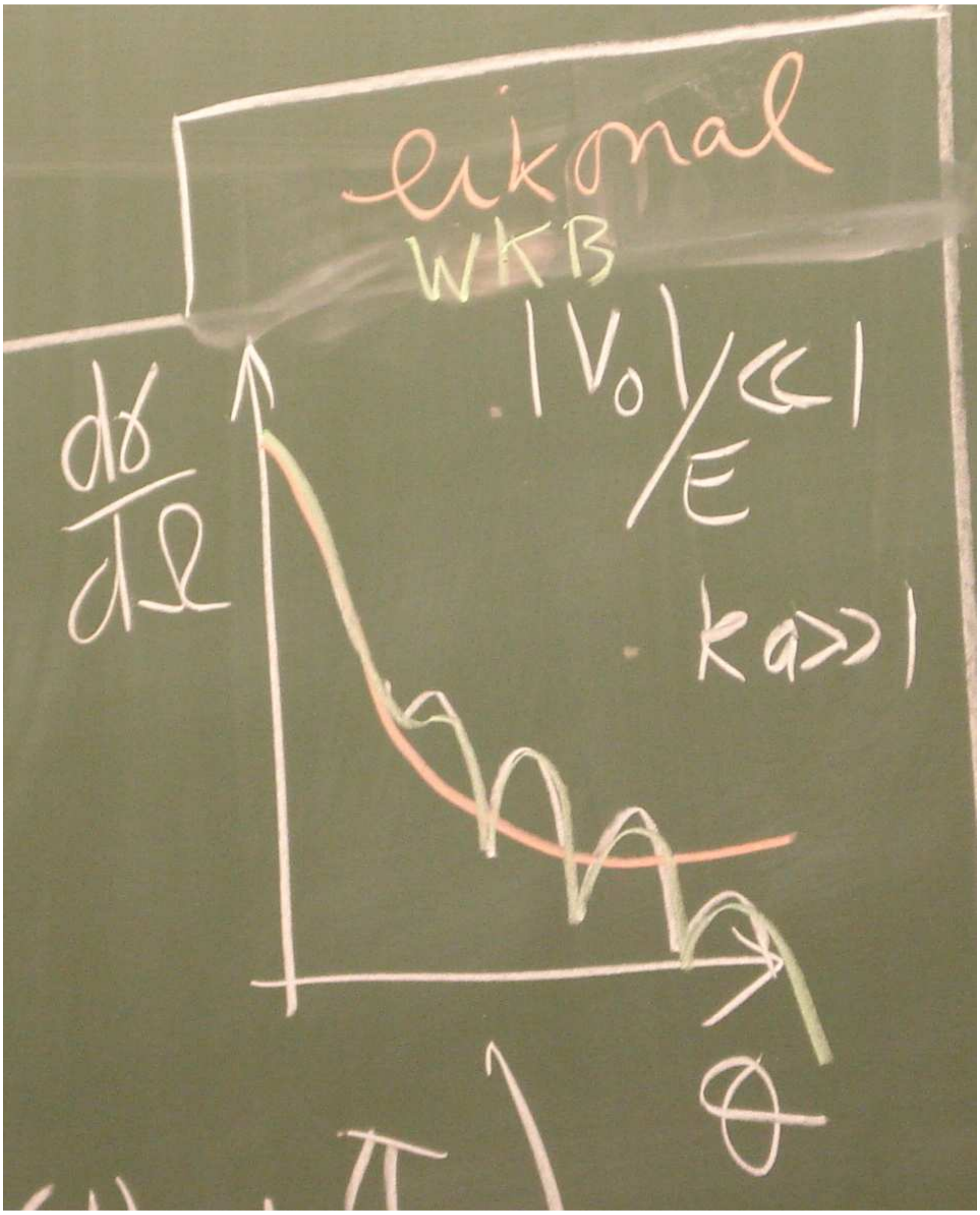

Figure 3. Red is eikonal, green is WKB. WKB is exact in the limit.

3 dimensions, radially symmetric potential

$$V(r) = -V_0\, r^{-\varepsilon}, \quad 1 < \varepsilon < 2, \quad V_0 > 0$$

(This excludes the Coulomb potential, unfortunately. That requires some additional tricks.)

Write the wavefunction as

$$\psi = \frac{1}{r} \sum_l c_l\, u_l(r)\, P_l(\cos\theta)$$

$$\left[\frac{\partial^2}{\partial r^2} + k^2 - \frac{2\,m}{\hbar^2} V(r) - \frac{l\,(l+1)}{r^2} \right] u_l(r) = 0$$

Sometimes, people use units where $m = \frac{1}{2}$ and $\hbar = 1$.

$$u_l(r) \sim \sin\left(k r - \frac{l\,\pi}{2} + \delta_l\right)$$

WKB gives

$$u_l(r) \sim \sin\left(\int_{r_l}^{r} \mathrm{d}r'\, k_l(r') - k\, r + \frac{\pi}{4}\right)$$

$$k_l(r) \neq \sqrt{k^2 - V(r) - \frac{l(l+1)}{r^2}}, \quad k_l(r) = \sqrt{k^2 - V(r) - \frac{(l+\frac{1}{2})^2}{r^2}}$$

(in this particular case, for this particular potential — correction due to singularity of potential).

$$\delta_l^{\mathrm{WKB}} = \lim_{r\to\infty}\left(\int_{r_l}^{r} \mathrm{d}r'\, k_l(r') - k\, r + \frac{\pi}{2}\left(l + \frac{1}{2}\right)\right)$$

1. Becomes exact when $|V_0|/E \ll 1$. $k\, a \gg 1$.
2. Insert δ_l^{WKB} into the partial wave expansion (1).

Rainbow scattering. Fraunhofer diffraction. Glory scattering. You name it, it can be worked out from this.

Under certain circumstances can obtain classical result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

3. Expand square root in $k_l(r)$. If you do that, you obtain the Eikonal phase shift.
4. Classical deflection function $\theta(E, b)$.

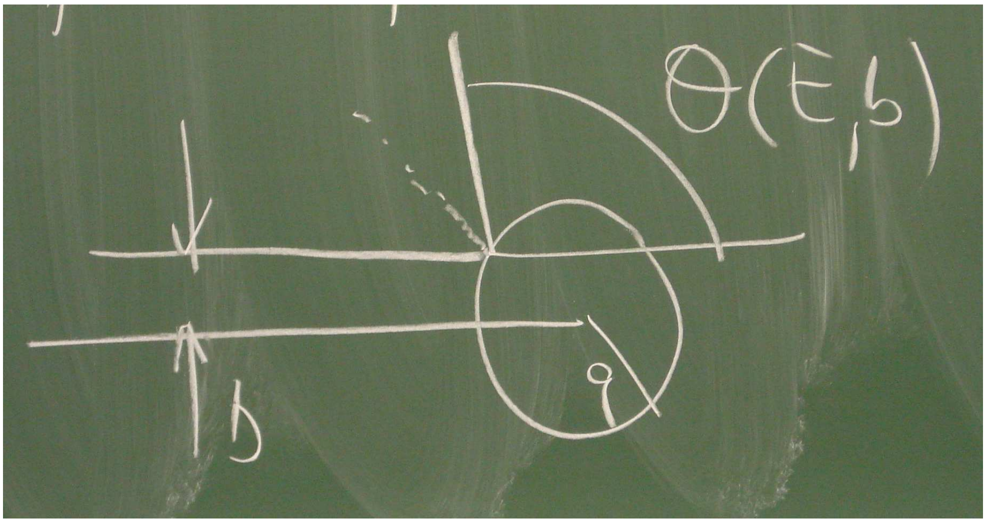

Figure 4.

$$\theta(E, b)\Big|_{b = l/k} = 2\,\frac{\partial}{\partial l}\,\delta_l^{\mathrm{WKB}}$$

Adiabatic quantum dynamics. Adiabatic here means slowly varying, as opposed to the meaning in thermodynamics. It is the evolution of quantum system under a slowly varying $\hat{H}(t)$.

Adiabatic theorem M.Born and V. Fock (1928).

If the system is in an instantaneous eigenstate of $\hat{H}(t)$ it will remain in this state if $\hat{H}(t)$ is sufficiently slowly varying.

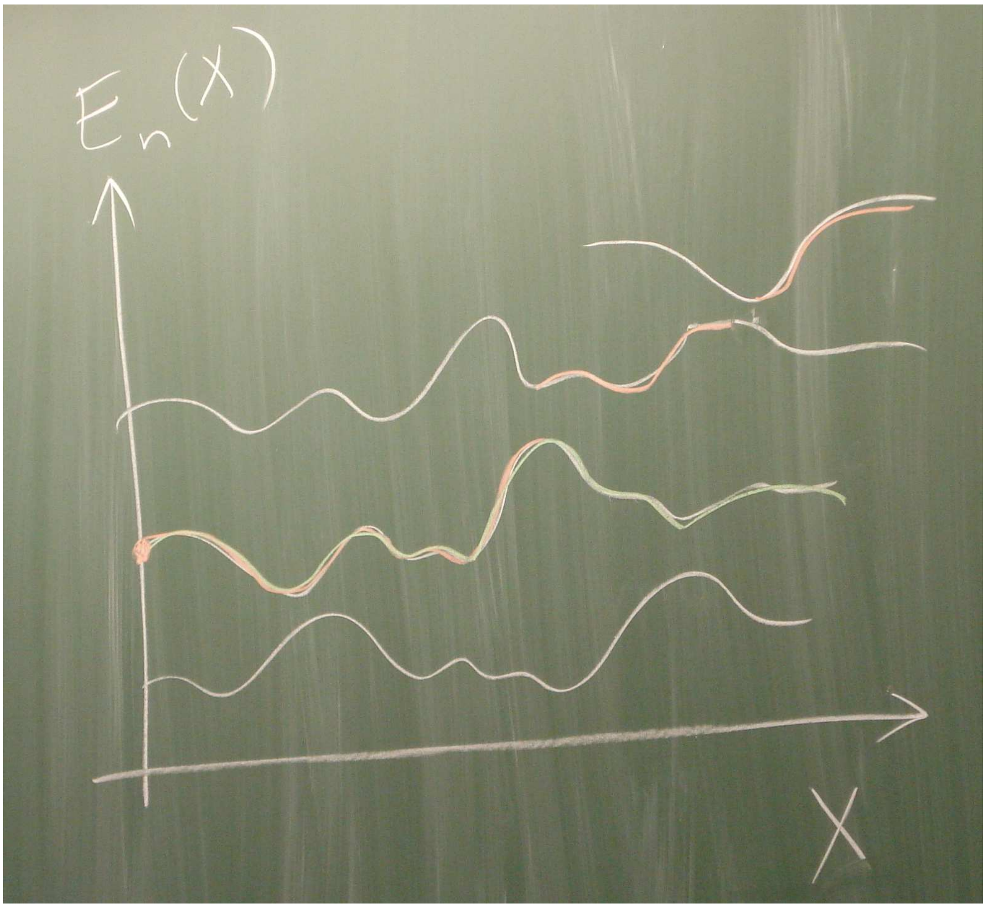

Figure 5. $E_n(x)$, x is perturbation.

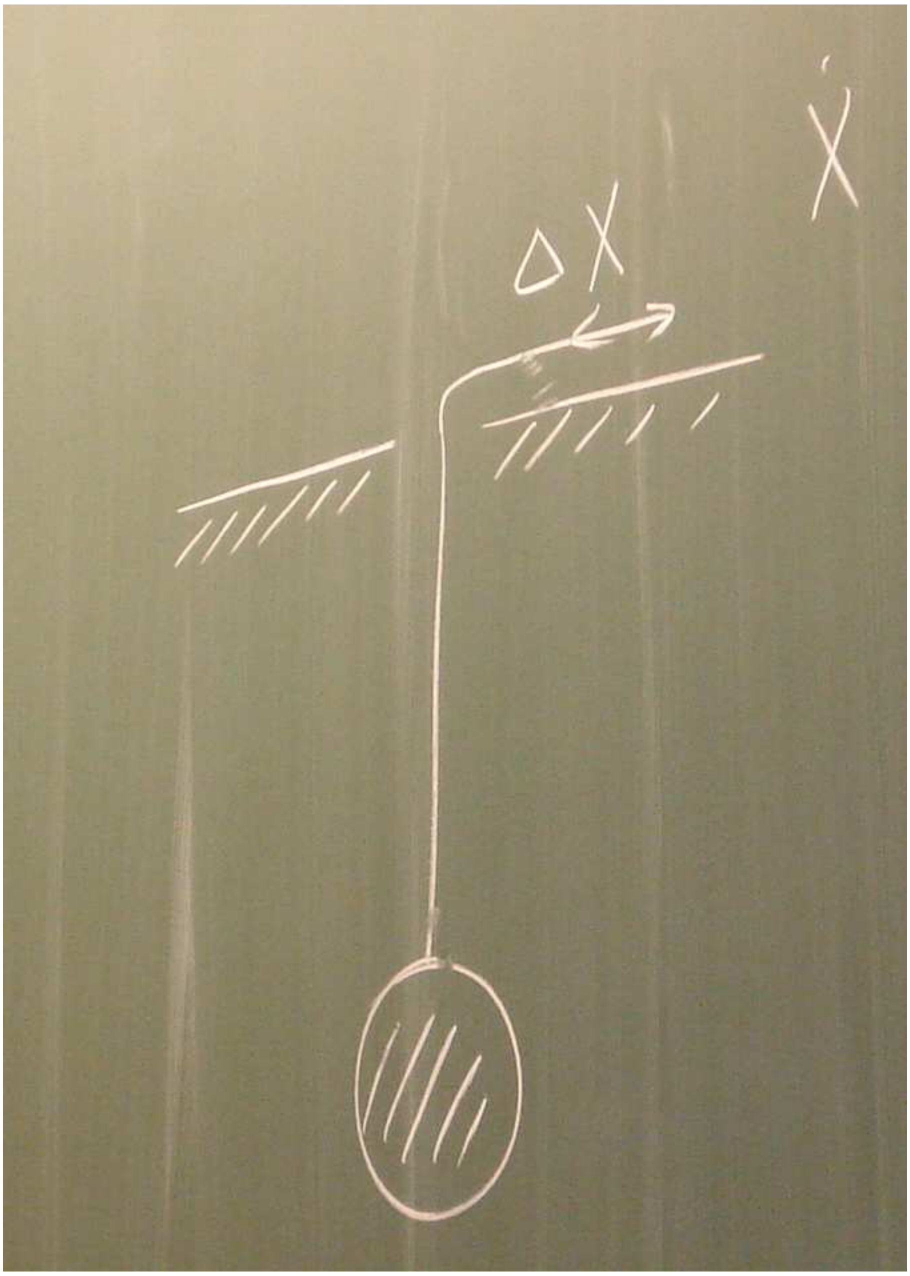

Figure 6. Pendulum with varying length.

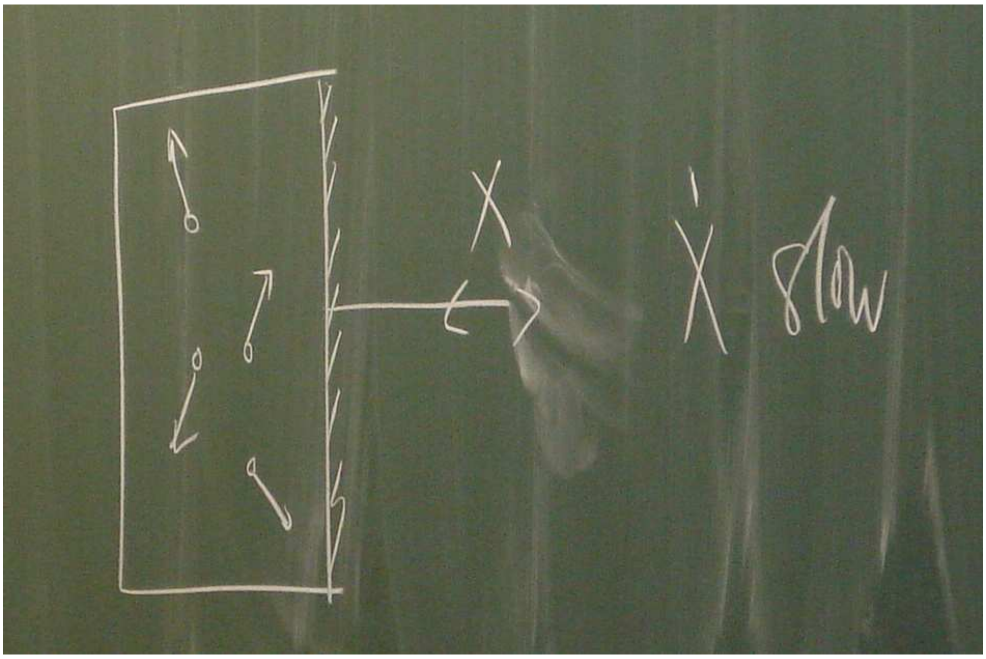

Figure 7. Gas of particles move randomly. Piston moves with $\dot{x}$ (slow).

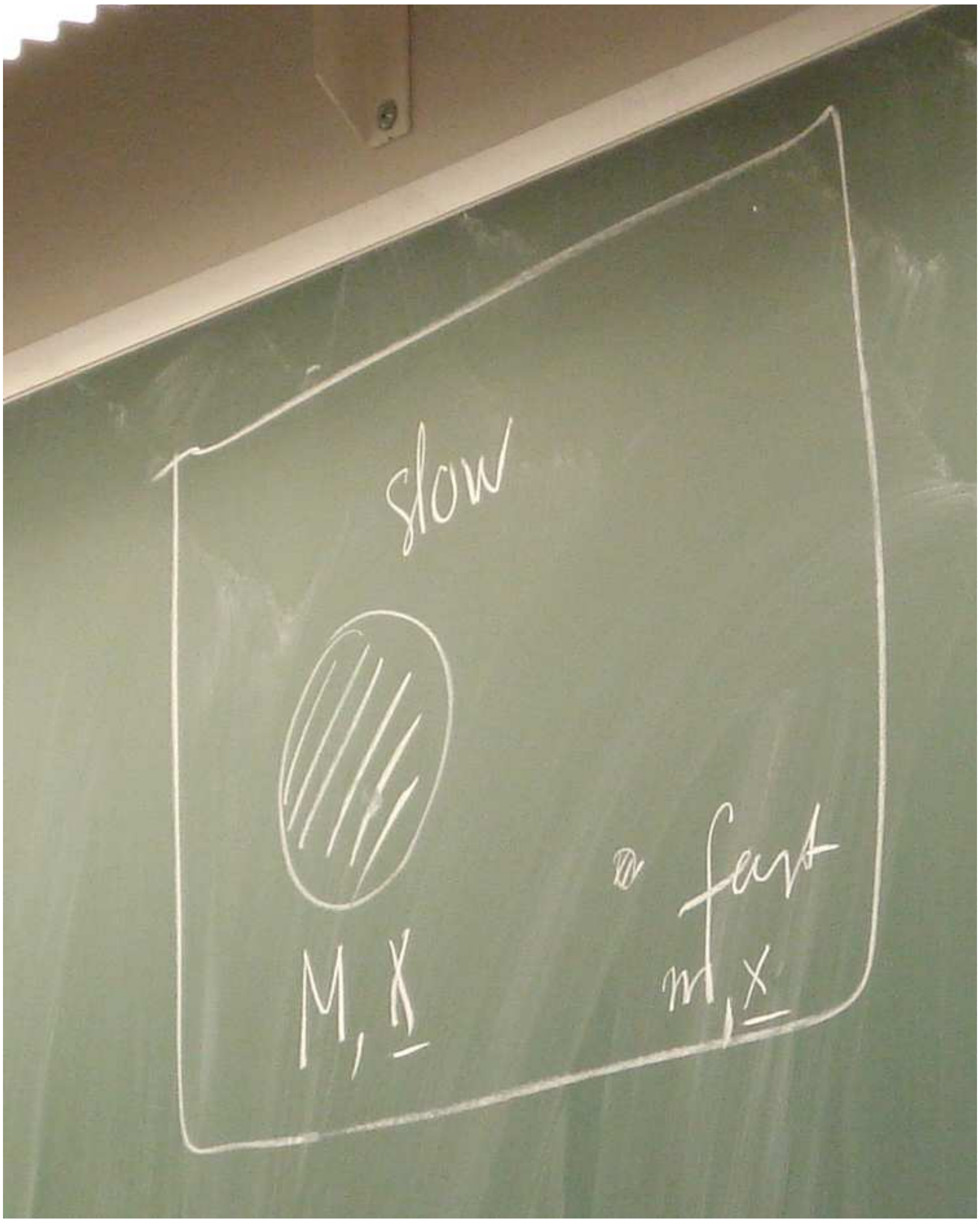

Figure 8. We study a slow, massive particle (mass M, position $\boldsymbol{X}$) and a faster, lighter particle (mass m, position $\boldsymbol{x}$), enclosed in a box.

$$\mathrm{i}\,\hbar\,\partial_t|\psi\rangle = \hat{H}\,(\boldsymbol{X}(t))|\psi\rangle$$

$$\hat{H}\,(\boldsymbol{X}(t)) = -\frac{\hbar}{2m}\frac{\partial}{\partial\boldsymbol{x}^2} + V(\boldsymbol{x}) + V(\boldsymbol{x},\boldsymbol{X}(t))$$

$$M\ddot{\boldsymbol{X}} = -\frac{\partial V_{\text{eff}}}{\partial X} = -\,\langle\psi|\frac{\partial\hat{H}}{\partial\boldsymbol{X}}|\psi\rangle$$

(Hellman–Feynman theorem). What you want to calculate is

$$\frac{\partial}{\partial X}\langle\psi|\hat{H}\,|\psi\rangle = \langle\psi|\frac{\partial}{\partial x}\hat{H}\,|\psi\rangle$$

Somehow, the change of $|\psi\rangle$ does not matter. That's the Hellman–Feynman theorem.

– The Born-Oppenheimer approximation. Molecular and solid-state physics.

– The absorption of infrared light by small metallic particles.

– Aharanov Bohm system. $\Phi(t)$. Time-dependent magnetic flux.

– Atoms, molecules, stuff generally, moving slowly through static fields.

– Atomic and molecular collisions.

– Landau-Zener theory.

– Berry's phase.

["And then this thing will jump and everything explodes."]