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A Lie group is a group that forms a manifold of finite dimension n, fulfilling the requirements

a) if $g\,g' = g''$, then the coordinates of g'' are smooth functions of the coordinates of g and g'.

Lie algebra = powerful tool for studying Lie gropus. Amounts to studying Lie group close to the identity element 1.

Definition: Lie algebra $\mathcal{G}$ of a Lie group G = tangent space of G at 1, with an extra structure, see below. This means, a group element g close to identity 1 can be written $g = 1 + \varepsilon\,A + \mathcal{O}(\varepsilon^2)$ where ε is a small number. Also, if $A \in \mathcal{G}$ then

$$\exp A = \mathrm{e}^A = \lim_{N\to\infty}\left(1 + \frac{1}{N}\,A\right)^N = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\,A^\nu \in G$$

(In all our examples g and A are $n \times n$ matrices, so pruducts of A's are well defined. Mathematicians define Lie algebras abstractly, have to do some more work here.)

Question: which Lie group elements can be obtained by exponentiation from $\mathcal{G}$?

If $A, B \in \mathcal{G} \Rightarrow G \ni \mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^C$ for some $C \in \mathcal{G}$? Look first at group elements close to 1.

$$\mathrm{e}^{\varepsilon\,A}\,\mathrm{e}^{\varepsilon\,B} = \mathrm{e}^{(\varepsilon)}?$$

To order ε^2?

$$\left(1 + \varepsilon\,A + \frac{1}{2}\,\varepsilon^2\,A^2\right)\left(1 + \varepsilon\,B + \frac{1}{2}\,\varepsilon^2\,B^2\right) = 1 + C + \frac{1}{2}\,C^2$$

$$1 + \varepsilon(A + B) + \frac{1}{2}\,\varepsilon^2\underbrace{(A^2 + 2\,A\,B + B^2)}_{(A+B)^2 + A\,B - B\,A} + \cdots$$

$$1 + \varepsilon(A + B) + \frac{1}{2}\,\varepsilon^2[A, B] + \cdots + \frac{1}{2}(\varepsilon(A + B) + \mathcal{O}(\varepsilon^2))^2 + \cdots$$

requires $C(\varepsilon) = \varepsilon(A + B) + \frac{1}{2}\,\varepsilon^2\,[A, B] + \mathcal{O}(\varepsilon^3)$. Conclusion $C \in \mathcal{G}$ requires $[A, B] \in \mathcal{G}$.

Mathematical definition of a Lie algebra $\mathcal{G}$

$\mathcal{G}$ is a vector space over some number field $\mathbb{F}$. In physics usually $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ (complex or real numbers) with an additional structure: the Lie bracket.

$A, B \in \mathcal{G} \Rightarrow [A, B] \in \mathcal{G}$.

In our matrix examples $[A,\, B] = A\;B - B\;A$ (commutator) but mathematicians define $[A,\, B]$ more abstractly. Need not be the commutator, but satisfies axioms. Is linear in A and in B and is antisymmetric $[A, B] = -[B, A]$; satisfies the Jacobi identity.

Remark: The Poisson bracket satisfies these axioms. $\Rightarrow$ There is a Lie algebra in Hamiltonian dynamics.

So, if $A, B \in \mathcal{G}$ then, by definition of $\mathcal{G}$

$$\mathrm{e}^{\varepsilon\,A}\,\mathrm{e}^{\varepsilon\,B} = \mathrm{e}^{C(\varepsilon)}$$

for some $C(\varepsilon) = g$ at least to order ε^2. Then according to a theorem, Baker-Hausdorff theorem, $C(\varepsilon) \in \mathcal{G}$ and can be written as an infinite sum of multicommutators of A and B. Explicit expression, first obtained by Dynkin. Our example groups:

$\mathbb{F}$	$\dim \mathcal{G}$	G	$\mathcal{G}$
$\mathbb{R}$	$\frac{1}{2}n(n-1)$	$\mathrm{O}(n)$	A is $n \times n$ real antisymmetric, $A + A^T = 0$
$\mathbb{R}$	n^2	$U(n)$	A is $n \times n$ complex antihermitian, $A + A^\dagger = 0$
$\mathbb{R}$	$n^2 - 1$	$\mathrm{SU}(n)$	in addition, $\mathrm{tr}(A) = 0$
$\mathbb{R}$	n^2	$\mathrm{GL}(n)$	A is $n \times n$ real arbitrary
$\mathbb{R}$	$n^2 - 1$	$\mathrm{SL}(n)$	A in addition traceless

In ll these examples elements of G $\mathcal{G}$ are $n \times n$ matrices, at as linear transformations in some n-dimensional vector space. One says that one has an n dimensional representation of G or $\mathcal{G}$ and the space of vectors $\boldsymbol{\varphi}$ is representation module $= V$. A group can have many different representations.

Adjoint representation. Choose basis for $\mathcal{G}$ $\{A_a, a = 1, \ldots, \dim \mathcal{G}\}$. Then $[A_a, A_b] = \sum_c f_{ab}{}^c A_c$, that is, commutation realtions are summarised by n^3 constants $f_{ab}{}^c$. Interpretation: $\mathcal{G}$ is represented on itself, $V = \mathcal{G}$. $A_a \in \mathcal{G}$ transforms $A_b \in V$ to $f_{ab}{}^c A_c \in V$. A_a is represented by a matrix $(M_a)^c{}_b =$ adjoint representation matrix.

Note: $[A_{a'}[A_{a'}, A_b]] = [A_{a'}, f_{ab}{}^c A_c] = f_{ab}{}^c f_{a'c}{}^d A_d = A_d (M_{a'})^d{}_c (M_a)^c{}_b$.

In order to check that this is really a representation of $\mathcal{G}$ one must check that the commutator of A_a and $A_{a'}$ is represented by the commutator of the matrices M; i.e.,

$$[[A_{a'}, A_a], A_b] = [A_{a'}, [A_a, A_b]] - [A_a [A_{a'}, A_b]]$$

but this is an identity.

Cartan-Killing form of a Lie algebra. Def by

$$g_{ab} = -\frac{1}{2}\mathrm{tr}(M_a M_b) = -\frac{1}{2}\sum_{c,d} f_{ac}{}^d f_{bd}{}^c$$

For albelian Lie albegra $f_{ab}{}^c = 0$ and $g_{ab} = 0$. On the other extreme among Lie algebras there are semisimple Lie algebras for which g_{ab} is nonsingular, and works as metric on $\mathcal{G}$. Is used to raise and lower indices, $g^{ab} = (g^{-1})^{ab}$. The analog in Lie algebra theory of the concept of invariant subgroups in group theory is called *ideal*.

Definition: $\mathcal{H} \subset \mathcal{G}$ is called an ideal of $\mathcal{G}$ if

1. $\mathcal{H}$ is itself a Lie algebra.

2. $A \in \mathcal{G}, B \in \mathcal{H}$ then $[A, B] \in \mathcal{H}$.

A Lie algebra $\mathcal{G}$ which has no ideals except $\{\mathbf{0}\}$ and $\mathcal{G}$ it is called simple.

Another property of semisimple Lie algebra. A Lie algebra which has no *albelian ideal* is called *semisimple*.

Same method as used on SU(2) in quantum mechanics can be generalized to arbitrary simple Lie algebras, all such, and all unitary irreducible representations, of them are classified:

1) Simple Lie algebras: $A_n, B_n, C_n, D_n, n \geqslant 1$ with some muntiple counting.

E_6, E_7, E_8, F_4, G_2.

2) All their irreducible representations, see Pope!