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Today procedure of gauging or Yang Mills theory

(Important in particle physics.)

Lecture 10:

$$\mathcal{L} = \sum_{k=1}^{n} \frac{1}{2}\left(\partial_\mu\varphi_k\,\partial^\mu\varphi_k - m^2\,\varphi_k\varphi_k\right)$$

In matrix notation:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\boldsymbol{\varphi}^T\partial^\mu\,\boldsymbol{\varphi} - m^2\boldsymbol{\varphi}^T\boldsymbol{\varphi}\right)$$

has $\mathrm{O}(n)$ symmetry: $\boldsymbol{\varphi}\to\boldsymbol{\varphi}' = O\,\boldsymbol{\varphi}$ where O is an $\mathrm{O}(n)$ matrix.

But $\mathcal{L}$ is invariant only if all rotation angles are the same everywhere in spacetime. From a general relativistic point of view, this is a bit unnatural. It would be more natural if $O = O(x,t) =$ arbitrary $\mathrm{O}(n)$ matrix function on spacetime. This can be arranged, by modifying the action. The procedure called "gauging the $\mathrm{O}(n)$ symmetry", or "making the $\mathrm{O}(n)$ symmetry local".

The mass term is already gauge invariant.

$$\boldsymbol{\varphi}^T\boldsymbol{\varphi}\to\boldsymbol{\varphi}^T O^T(x)\,O(x)\,\boldsymbol{\varphi} = \boldsymbol{\varphi}^T\boldsymbol{\varphi}$$

The problem is the kinetic term, the time derivative terms:

$$\partial_\mu\boldsymbol{\varphi}\to\partial_\mu\left(O(x)\,\boldsymbol{\varphi}\right) = O_{,\mu}\,\boldsymbol{\varphi} + O\,\partial_\mu\boldsymbol{\varphi}$$

Here we have a problem term, $O_{,\mu}\boldsymbol{\varphi}$.

Remedy: Introduce an $n\times n$ matrix of vector fields B_μ, transforming under $\mathrm{O}(n)$ gauge transformations as $B_\mu\to B'_\mu = O\,B_\mu\,O^{-1} + O_{,\mu}O^{-1}$ and define covariant derivative D_μ:

$$D_\mu\boldsymbol{\varphi}\equiv(\partial_\mu - B_\mu)\boldsymbol{\varphi} \quad\Rightarrow\quad D_\mu\,\boldsymbol{\varphi}\to O_{,\mu}\boldsymbol{\varphi} + O\,\partial_\mu\boldsymbol{\varphi} - \left(O\,B_\mu O^{-1} + O_{,\mu}O^{-1}\right)O\varphi = O\,(\partial_\mu\varphi - B_\mu\varphi) =$$

$$= O\,D_\mu\,\varphi$$

$$\Rightarrow (D_\mu\,\varphi)^T(D^\mu\varphi)\text{ is gauge invariant}$$

So we have managed to make $\mathcal{L}$ gauge invariant, now $\mathcal{L} = \frac{1}{2}(D_\mu\varphi)^T(D^\mu\varphi) - \frac{1}{2}m^2\varphi^T\varphi$.

Remark: In this example $O^{-1} = O^T$ and $0 = \partial\left(O\,O^{-1}\right) = O_{,\mu}O^{-1} + O\,(O^{-1})_{,\mu}$.

$$(B'_\mu)^T = (O\,B_\mu\,O^{-1} + O_{,\mu}O^{-1})^T \quad\Rightarrow\quad \left(B'_\mu\right)^T = O\,B^T O^{-1} - O_{,\mu}\,O^{-1}$$

B_μ can be chosen antisymmetric $n\times n$ matrix.

Remark: In this example the symmetry was $\mathrm{O}(n)$, but the same procedure works for any continuous symmetry.

Example: Suppose ψ is a complex n-component vector field and $\mathcal{L} = \partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi \equiv \sum_k\left(\partial_\mu\psi_k^*\,\partial^\mu\psi_k - m^2\psi_k^*\,\psi_k\right)$. Then $\mathcal{L}$ is invariant under unitary transformations.

$$\psi\to\psi' = U\,\psi$$

where U is a unitary $n \times n$ matrix, $U^\dagger U = 1$. Same procedure, gauging, applies in this case, except now the matrix B_μ should be chosen antihermitian $B_\mu^\dagger = -B_\mu$.

Back to the orthogonal case. In $\mathcal{L}(\varphi, \varphi_{,\mu}, B_\mu)$ the B_μ occurs only undifferentiate, is therfore not dynamical. But it is possible to add gauge invariant kinetic energy terms to $\mathcal{L}$.

We had: $\varphi \to O\,\varphi$, $D_{,\mu}\varphi \to O\,D_\mu \varphi = O\,D_\mu\,O^{-1}(O\,\varphi)$. Hence $D_\mu \to O\,D_\mu\,O^{-1}$.

$$\left(D_\mu \to O\,D_\mu\,O^{-1} \to O\,O' D_\mu\,O'^{-1} O^{-1} = (O\,O')D_\mu\,(O\,O')^{-1}\right)$$

But note that the derivative operator here acts on O^{-1} and then further to the right.

$$\Rightarrow D_\mu D_\nu - D_\nu D_\mu \equiv [D_\mu, D_\nu] \to O\,[D_\mu, D_\nu]\,O^{-1}$$

but here no derivatives act on O^{-1}.

$$[D_\mu, D_\nu] = [\partial_\mu - B_\mu, \partial_\nu - B_\nu] = -\,(\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu]) = -\,G_{\mu\nu}$$

$$\begin{aligned} B_\mu &= \text{gauge field potential} \\ G_{\mu\nu} &= \text{gauge field strength} \end{aligned}$$

Lorentz invariant object with B derivatives:

$$[D_\mu, D_\nu][D^\mu, D^\nu] = G_{\mu\nu} G^{\mu\nu} \to O\,G_{\mu\nu} G^{\mu\nu}\,O^{-1}$$

$\mathrm{tr}([D_\mu,\ D_\nu][D^\mu,\ D^\nu])$ is gauge invariant and Lorentz invariant and can be added to the Lagrangian. Then fianlly one arrives at the Lagrangian density

$$\mathcal{L} = \frac{1}{8\,g^2}\,\mathrm{tr}(G_{\mu\nu} G^{\mu\nu}) + \frac{1}{2}\,(D_\mu\varphi)^T (D^\mu\varphi) - \frac{1}{2}\,m^2 \varphi^T \varphi$$

This is called Lagrangian for an O(n) Yang-Mills theory.

Example: $n = 2$.

$$B_\mu = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} A_\mu = E\,A_\mu$$

where E is the 2×2 matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and A_μ is a vector field.

$$G_{\mu\nu} = (\partial_\mu\,A_\nu - \partial_\nu\,A_\mu)E = F_{\mu\nu}\,E$$

$$\frac{1}{8\,g^2}\,\mathrm{tr}(G_{\mu\nu}\,G^{\mu\nu}) = -\,\frac{1}{4\,g^2}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{-1}{4\,g^2} F_{\mu\nu} F^{\mu\nu}$$

$$\frac{1}{2}(D_\mu\,\varphi)^T (D^\mu\,\varphi) = \frac{1}{2}\left(\varphi_{,\mu}^T + \varphi^T E\,A_\mu\right) \times (\varphi'^\mu - E\,A^\mu \varphi) =$$

$$= \frac{1}{2}\,\partial_\mu \varphi^T \partial^\mu \varphi - \varphi_{,\mu}^T E\,\varphi\,A^\mu + \frac{1}{2}\,A_\mu A^\mu \varphi^T \varphi$$

This $\mathcal{L}$ describes electrodynamics with charged scalar field instead of charged particles, and the electromagnetic current = the Nöther current of the O(2) symmetry.

Remember $\delta\varphi = \varepsilon\, E\, \varphi$ and $\mathcal{L}(\varphi, \varphi_{,\mu}, A_\mu)$ invariant:

$$j_\mu = \Pi_\mu\ E\,\varphi = (D_\mu \varphi)^T E\,\varphi = \left(\varphi^T_{,\mu} + \varphi^T E\, A_\mu\right) E\,\varphi =$$

$$= \varphi^T_{,\mu} E\,\varphi - \varphi^T \varphi\, A_\mu$$

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = \frac{\partial \mathcal{L}}{\partial A_\mu}$$

$$\frac{1}{g^2}\, \partial_\nu F^{\mu\nu} = -\, j^\mu$$

More standard notation: $F \to g\, F$, $g \to e$.

Definition: set of all $\mathrm{O}(n)$ transformation, i.e. set of all orthogonal $n \times n$ matrices, $O^T O = 1$ is called the orthogonal group in n dimensions, denoted $\mathrm{O}(n)$.

Mathematical definition of a group:

A group G is a set with the properties

1. A composition law, $\circ$, which to any ordered pair of elements in G, $a \in G$, $b \in G$ associate a third element $c = a \circ b$ (the product of a and b).

2. $a \circ (b \circ c) = (a \circ b) \circ c$: the associative law.

3. G contains a unique unit element e such that $\forall\, a \in G$: $e \circ a = a \circ e = a$.

4. $\forall a \in G, \exists$ unique $a^{-1} \in G$, such that $a \circ a^{-1} = a^{-1} \circ a = e$.