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Addition to general analytic mechanics

You know that conservation laws help solving equations of motion. Consider a system with Hamiltonian $H(p_1, \ldots, p_n, q^1, \ldots, q^n)$.

Theorem: The system is *integrable* iff there are n conservation laws $A_k(p, q)$, $k = 1, \ldots, n$, with zero Poisson brackets. $[A_k, A_l]_{\mathrm{PB}} = 0$ for all k, l. A solution to the equations of motion $q^\nu(t)$, $p_\nu(t)$, $t_0 < t < \infty$ then lies on an n-dimensional submanifold of phase space, given by $A_k(p, q) =$ constants $= A_k(t = t_0)$.

Example: $n = 2$. Particle in a central potential.

$$L = \frac{1}{2}\left(m\,\dot{r}^2 + m\,(r\,\dot{\theta})^2\right) - V(r)$$

$$p_r = m\,\dot{r}\,, \quad p_\theta = m\,r^2\,\dot{\theta}$$

$$H(p_r, p_\theta, r, \theta) = \frac{1}{2m}\,p_r^2 + \frac{1}{2\,m\,r^2}\,p_\theta^2 + V(r)$$

H, p_θ conserved $\Rightarrow$integrable.

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$H = \frac{1}{2m}\,p_r^2 + \frac{1}{2\,m\,r^2}\,p_\theta^2 + V(r) = E$$

can be solved for $p_r = p_r(r, p_\theta, E)$.

$$\dot{r} = \frac{1}{m}\,p_r(r, p_\theta, E)$$

This is a separable differential equation.

$$\int \frac{\mathrm{d}r}{p_r/m} = \int \mathrm{d}t \quad \Rightarrow \quad r(t)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\,r^2(t)} \quad \Rightarrow \quad \theta = \theta(t)$$

Integrable: the solution of the differential equations are reduced to simple integrals.

Conversely, it is known that for nonintegrable systems, a solution to the equations of motion is not so restricted, to an n dimensional submanifold of phase space. Also, the solutions to the equations of motion behave very differently from the integrable case. Chaotic behaviour.

Example: It is not known whether the solar system is stable (mathematically, as a nine particle system), or if e.g. the earth is expelled after some 10^{10} years.

Topics omitted in this course:

— Hamilton-Jacobi theory.

— Action-angle variables.

— Canonical perturbation theory.

— Properties of the equations of motion.

Solving system of linear differential equations with constant coefficients, see Goldstein, small oscillations, chapter 6.

Example:

$$A\,\ddot{x} + B\,\dot{x} + C\,x = 0$$

A: mass; B: damping coefficient; C: spring constant.

Ansatz: $x = a\,\mathrm{e}^{\mathrm{i}\omega t}$

$$\Rightarrow (-\omega^2\,A + \mathrm{i}\,\omega\,B + C)\,a\,\mathrm{e}^{\mathrm{i}\omega t} = 0$$

$$a \neq 0 \quad \Rightarrow \quad -\omega^2\,A + \mathrm{i}\,\omega\,B + C = 0$$

Then $x(t) = a_1\,\mathrm{e}^{\mathrm{i}\omega_1 t} + a_2\,\mathrm{e}^{\mathrm{i}\omega_2 t}$ solves the differential equation and contains two arbitrary parameters. It is therefore the general solution. (If $\omega_1 = \omega_2$ we get a polynomial factor $x(t) = (a + t\,b)\,\mathrm{e}^{\mathrm{i}\omega t}$.

System of such equations are solved the same way.

$$\sum_{j=1}^{n}\,(A_{ij}\,\ddot{x}_j + B_{ij}\,\dot{x}_j + C_{ij}\,x_i) = 1, \quad i = 1, \ldots, n$$

Ansatz: $x_j = a_j\,\mathrm{e}^{\mathrm{i}\omega t}$.

$$\sum_j\,((-\omega^2)A_{ij} + \mathrm{i}\,\omega\,B_{ij} + C_{ij})a_j = 0$$

This is a linear system of n equations for $\boldsymbol{a}$ = vector in n dimensional space. Solution $\boldsymbol{a} \neq \boldsymbol{0}$ requires that the matrix is singular:

$$\det\,(-\omega^2\,A + \mathrm{i}\,\omega\,B + C) = 0$$

This is an equation of degree $2n \Rightarrow 2n$ roots. $\omega = \omega_k, k = 1, \ldots, 2\,n$. For each ω_k find $\boldsymbol{a}_k \neq 0$ which solves the equation. General solution:

$$x_j(t) = \sum_k\,\alpha_k\,a_{jk}\,\mathrm{e}^{\mathrm{i}\omega_k t}$$

$2\,n$ arbitrary parameters $\alpha_k \Rightarrow$ this is the general solution.

Remark. The system can be Hamiltonian with Lagrangian

$$L = \sum_{ij}\,\frac{1}{2}(\dot{x}_i\,A_{ij}\,\dot{x}_j + x_i\,B_{ij}\,\dot{x}_j - x_i\,C_{ij}\,x_j)$$

If A and C are symmetric, B antisymmetric.

Continuation from friday

System of n free scalar fields $\varphi_k(x), k = 1, \ldots, n$.

$$\mathcal{L} = \frac{1}{2}\,\sum_k\,(\partial_\mu\,\varphi_k\,\eta^{\mu\nu}\partial_\nu\varphi_k - m^2\,\varphi_k\,\varphi_k)$$

⇒Energy momentum tensor

$$j^{\mu}{}_{\nu} = \sum_k \frac{\partial \mathcal{L}}{\partial \varphi_{k,\mu}} \varphi_{k,\mu} - \delta^{\mu}_{\nu} \mathcal{L}$$

But now there is also another symmetry. Regard $\varphi_k(x)$ as components of a vector $\boldsymbol{\varphi}(x)$ (in internal n dimensional space). $\mathcal{L}$ is invariant under (orthogonal) rotations in this space, an $\mathrm{O}(n)$ symmetry.

In matrix notation:

$$\varphi(x) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} \quad \mapsto \quad \varphi'(x) = \mathcal{O}\, \varphi(x)$$

where $\mathcal{O}$ is an orthogonal $n \times n$ matrix. $\mathcal{O}^T \mathcal{O} = 1$. E.g.

$$\sum_k \varphi_k \varphi_k = \varphi^T \varphi \quad \rightarrow \quad (\mathcal{O}\varphi)^T \mathcal{O}\varphi = \varphi^T \mathcal{O}^T \mathcal{O}\varphi = \varphi^T \varphi$$

$$\mathcal{O} = 1 + \varepsilon\, A$$

$$\mathcal{O}^T \mathcal{O} = (1 + \varepsilon\, A)^T (1 + \varepsilon\, A) = 1 + \varepsilon\, (A^T + A) + O(\varepsilon^2)$$

$$A^T + A = 0$$

A is antisymmetric. Now $\mathcal{L}$ invariant, $g = 0, \delta\varphi = \varepsilon\, A\, \varphi$.

$$j_A{}^{\mu} = \sum_k \frac{\partial \mathcal{L}}{\partial \varphi_{k,\mu}}\, \delta\varphi = \eta^{\mu\nu} \varphi^T_{,\nu}\, A\, \varphi$$

$$Q_A = \int \mathrm{d}^3 x\, \dot{\varphi}^T A \varphi$$

$$\text{Equations: } \dot{Q}_A = 0$$

In Hamiltonian form

$$Q_A = \int \mathrm{d}^3 x\, \Pi^T A \varphi$$

There are $\frac{1}{2}\, n\, (n - 1)$ linearly independent such conserved charges, one for each antisymmetric matrix.

Action for the electro-magnetic field. NB. Units such that $c = 1$.

Remember particle action:

$$\int (-m\, c\, \mathrm{d}s + q\, (\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} - \varphi\, \mathrm{d}t))$$

For n charged particles, each with its curve parameter $s_k, k = 1, \ldots, n$ and world line $x^{\mu}(s_k) = x^{\mu}_k$.

$$-q \int A_{\mu}\, \mathrm{d}x^{\mu} \quad \rightarrow \quad -\sum_k q_k \int \mathrm{d}s_k\, A_{\mu}(x(s_k))\, \dot{x}^{\mu}(s_k) = -\int \mathrm{d}^4 x\, A_{\mu}(x)\, j^{\mu}(x) = A_{\text{int}}$$

$$j^{\mu}(x) = \sum_k q_k \int \mathrm{d}s_k\, \delta^4(x - x(s_k)) \dot{x}^{\mu}(s_k)$$

Maxwell equations

$$\begin{array}{lll} \nabla\times\boldsymbol{B}-\dot{\boldsymbol{E}}=\boldsymbol{j} & \nabla\cdot\boldsymbol{B}=0 & \boldsymbol{B}=\nabla\times\boldsymbol{A} \\ \nabla\cdot\boldsymbol{E}=\rho & \nabla\times\boldsymbol{E}+\dot{\boldsymbol{B}}=0 & \boldsymbol{E}=-\nabla\phi-\dot{\boldsymbol{A}} \end{array}$$

$$\nabla\times(\nabla\times\boldsymbol{A})+\nabla\dot{\phi}+\ddot{\boldsymbol{A}}=j=\frac{\partial\mathcal{L}_{\text{int}}}{\partial\boldsymbol{A}}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}_{\text{EM}}}{\partial\dot{\boldsymbol{A}}}+\ldots$$

$$-\nabla^2\phi-\nabla\cdot\dot{\boldsymbol{A}}=\rho=-\frac{\partial\mathcal{L}_{\text{int}}}{\partial\phi}=-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}_{\text{EM}}}{\partial\dot{\varphi}}+\ldots$$

$$A_{\text{int}}=\int \mathrm{d}^4x\,(\boldsymbol{A}\cdot\boldsymbol{j}-\phi\,\rho)$$

$$A=A_{\text{part}}+\underbrace{A_{\text{int}}+A_{\text{field}}}_{\int \mathrm{d}^4x(\boldsymbol{A}\cdot\boldsymbol{j}-\phi\rho+\mathcal{L}_{\text{EM}})}$$

$$\mathcal{L}_{\text{EM}}=\frac{1}{2}\big(\dot{\boldsymbol{A}}^2-(\nabla\times\boldsymbol{A})^2\big)+\dot{\boldsymbol{A}}\cdot\nabla\phi+\frac{1}{2}(\nabla\phi)^2=\frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{B}^2)$$

Note: Energy of field is $\frac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)$. $\frac{1}{2}\boldsymbol{E}^2=$ kinetic energy, $\frac{1}{2}\boldsymbol{B}^2=$potential energy.

4-dimensional notation:

$$x^\mu=(x^0,\boldsymbol{x})=(t,\boldsymbol{x})$$

$$x_\mu=\eta_{\mu\nu}x^\nu=(t,-\boldsymbol{x})$$

$$\partial_\mu=(\partial_t,\nabla),\qquad \partial^\mu=\eta^{\mu\nu}\partial_\nu=(\partial_t,-\nabla)$$

$$A^\mu=(A^0,\boldsymbol{A})=(\phi,\boldsymbol{A}),\quad A_\mu=\eta_{\mu\nu}A^\nu=(\varphi,-\boldsymbol{A})$$

(NB. Change of notation from lecture 8.)

One defines

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

$$F_{\mu\nu}=\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

Then $\frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{B}^2)=-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ is a Lorentz invariant.

$$\mathcal{L}_{\text{EM}}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu},\quad \mathcal{L}_{\text{int}}=-A_\mu j^\mu$$