2008–09–12

Variational calculus

Example of a problem in variational calculus:

Determine the shape of a soap bubble surface between two circular rings.

Solution: Choose cylindrical coordinates ρ, φ, z. The surface is described by $\rho = \rho(\varphi, z) = \rho(z)$. The rings sit at $z = z_1$, $\rho = a$ and $z = z_2$, $\rho = a$. The physical principle: area is minimized.

$$A[\rho] = \int_{z_1}^{z2} \mathrm{d}z\, 2\pi\rho(z)\sqrt{1 + (\rho'(z))^2}$$

with boundary conditions $\rho(z_1) = a$, $\rho(z_2) = a$. Variational calculus is a method to solve problems like: Given function F of three variables, find a function $\rho\colon [z_1, z_2] \to \mathbb{R}$ that minimizes

$$A[\rho] \equiv \int_{z_1}^{z_2} F(\rho(z), \rho'(z), z)\,\mathrm{d}z$$

with given boundary conditions $\rho(z_1) = \rho_1$, $\rho(z_2) = \rho_2$.

Compare with ordinary differential calculus: in the similar problem a function $f(x_1, \ldots, x_n)$ is minimized. The correspondence is

$$\begin{array}{rcl} A & \leftrightarrow & f \\ \rho(z) & \leftrightarrow & x_i \\ z & \leftrightarrow & i \\ [z_1, z_2] & \leftrightarrow & \{1, \ldots, n\} \end{array}$$

Reduce the problem to differential calculus as follows:

Suppose $\rho_0(z)$ gives the minimum. Consider another function $\rho(z) = \rho_0(z) + \Delta\rho(z)$ which satisfies the boundary conditions, but is otherwise arbitrary.

Consider the set of functions $\rho_\epsilon(z) = \rho_0 + \epsilon\,\Delta\rho$, where ϵ ranges over an interval around zero. $A(\epsilon) = A[\rho_\epsilon]$ has a minimum at $\epsilon = 0$.

$$\left.\frac{\mathrm{d}A(\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = 0$$

$$A(\epsilon) - A(0) = \int_{z_1}^{z_2} \mathrm{d}z \left(\frac{\partial F}{\partial\rho_0}\,\epsilon\,\Delta\rho + \frac{\partial F}{\partial\rho'}\,\epsilon\,\frac{\mathrm{d}}{\mathrm{d}z}(\Delta\rho)\right) + \mathcal{O}(\epsilon^2)$$

$$= \epsilon\left[\frac{\partial F}{\partial\rho_0'}\,\Delta\rho\right]_{z_1}^{z_2} + \epsilon\int_{z_1}^{z_2} \mathrm{d}z \left(\frac{\partial F}{\partial\rho_0} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial F}{\partial\rho'}\right)\Delta\rho + \mathcal{O}(\epsilon^2) =$$

$\Delta\rho(z_1) = 0, \Delta\rho(z_2) = 0$:

$$= \epsilon\int_{z_1}^{z_2} \mathrm{d}z \left(\frac{\partial F}{\partial\rho_0} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial F}{\partial\rho'}\right)\Delta\rho + \mathcal{O}(\epsilon^2)$$

$$0 = \left.\frac{\partial A(\epsilon)}{\partial\varepsilon}\right|_{\epsilon=0} = \int_{z_1}^{z_2} \mathrm{d}z \left(\frac{\partial F}{\partial\rho_0} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial F}{\partial\rho'}\right)\Delta\rho$$

$$\frac{\partial F}{\partial\rho_0} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial F}{\partial\rho'} = 0 \text{ for } z \in [z_1, z_2].$$

Same as Lagrange's equations in mechanics, with $F \to L$, $\rho \to q$, $z \to t$.

So, Lagrange's equations $\Rightarrow$**Hamilton's principle:**

A mechanical system moves between two times such that the **action** is stationary under small variations of the path such that the endpoints remain fixed:

$$\text{Action} = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t$$

Note: Usually the stationary point is a saddle point (neither max nor min).

Note: One advantage of Hamilton's principle: It's a simply stated complete characterisation of possible motions of a system, manifestly independent of the choice of coordinate.

Note: In practice, in variational calculations, one is more brief: One writes

$$\delta A[q] = \delta \int_{t_1}^{t_2} L(q, \dot{q}) \mathrm{d}t = \int \left(\frac{\partial L}{\partial q}\, \delta q + \frac{\partial L}{\partial \dot{q}}\, \delta \dot{q} \right) \mathrm{d}t = \left[\frac{\partial L}{\partial \dot{q}}\, \delta q \right]_{t_1}^{t_2} + \int \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q\, \mathrm{d}t$$

Note: System with holonomic constraints:

$$\frac{\partial L}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^\nu} + \sum_\alpha \lambda_\alpha(t) \frac{\partial f_\alpha}{\partial q^\nu} = 0, \quad \text{constraints } f_\alpha(q, t) = 0$$

Described by: action

$$A[q, \lambda] = \int \left(L(q, \dot{q}) + \sum_\alpha \lambda_\alpha f_\alpha(q) \right) \mathrm{d}t$$

Here $\lambda = \lambda(t)$ is a function of time.

Remark: In variational calculus Lagrange parameters which are not functions of time can also occur.

Example: Homogeneous chain. ⟨...⟩

Note: There do exist velocity dependent holonomic constraints, i.e. constraints $f_\alpha(q,\, \dot{q})$ which can be treated by forming L^{ext}.

I know no general method to decide from looking at a time dependent constraint whether it is hononomic or not.

Example illustrating the importance of boundary conditions.

Determine the shape of an elastic homogeneous rod fastened at its ends.

Solution:

Physics: potential energy is minimized. Describe shape by function $y(x), 0 \leqslant x \leqslant l$.

$$A[y] = V_{\text{gravitational}} + V_{\text{elastic}} = \int_0^l \mathrm{d}x \left[\rho\, g\, y(x) + \frac{1}{2} \kappa \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \right)^2 \right]$$

(if the rod is approximately horizontal). Compare Hook's law and the engergy of a spring. ρ is mass per unit length, κ is the stiffness of the rod.

$$\delta A = \int_0^l (\rho g \delta y + \kappa\, y''\, \delta y')$$

$$\kappa y'' \delta y'' = \kappa \frac{\mathrm{d}}{\mathrm{d}x} (y'' \delta y') - \kappa y''' \delta y'$$

$$-\kappa y''' \delta y' = -\frac{\mathrm{d}}{\mathrm{d}x} (\kappa\, y'''\, \delta y) + \kappa\, y^{\mathrm{IV}}\, \delta y$$

$$\delta A = \kappa [y''\, \delta y' - y''' \delta y]_0^l + \int_0^l \mathrm{d}x (p\, g + \kappa\, y^{\mathrm{IV}}) \delta y = 0$$

$\Rightarrow$1) Eqn: $pg+\kappa\, y^{\mathrm{IV}}=0, 0\leqslant x\leqslant l$

2) Boundary conditions.

a) $\delta y(0)=0$, $\delta y(l)=0$, but $\delta y'$ arbitrary. The boundary terms in δA require that $y''(0)=0$, $y''(l)=0$. $\Rightarrow$ in total 4 boundary conditions. Good, because the equation is of fourth order. We get four integration constants.

b) $\delta y(0)=0$, $\delta y'(0)=0$. No restriction on the other end. Boundary terms require $y''(l)=0$, $y'''(l)=0$. Again, in total 4 boundary conditions.