2008–09–05

Today: Lagranges equations.

Newtons equations $m\,\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$, where $i = 1, \ldots, N$. Choose Cartesian coordinates. The complete equations are

$$\begin{cases} m_i\ddot{x}_i &=& F_{x_i} \\ m_i\ddot{y}_i &=& F_{y_i}\,, \quad i=1,\ldots,N \\ m_i\ddot{z}_i &=& F_{z_i} \end{cases}$$

We change the notation and write the equations as $m_i\,\ddot{x}_i = F_i$ with $i = 1, \ldots, 3N \equiv n$ in three dimensions ($i = 1, \ldots, 2N$ in two dimensions). n is the number of degrees of freedom.

Make a general change of coordinates: $x \mapsto q$

$$x_i = x_i(q^1, \ldots, q^n, t), \quad i = 1, \ldots, n$$

$$\dot{x}_i = \frac{\mathrm{d}}{\mathrm{d}t}\,x_i = \sum_{\nu=1}^{n} \frac{\partial x_i}{\partial q^\nu}\,\dot{q}^\nu + \frac{\partial x_i}{\partial t} = \dot{x}_i(q, \dot{q}, t)$$

$$(\text{and}\ \mathrm{d}x_i = \sum_\nu \frac{\partial x_i}{\partial q^\nu}\,\mathrm{d}q^\nu + \frac{\partial x_i}{\partial t}\,\mathrm{d}t)$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}^\nu} = \frac{\partial x_i}{\partial q^\nu}$$

$$\frac{\partial \dot{x}_i}{\partial q^\nu} = \sum_\mu \frac{\partial^2 x_i}{\partial q^\nu\,\partial q^\mu}\,\dot{q}^\mu + \frac{\partial^2 x_i}{\partial q^\nu\,\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial x_i(q,t)}{\partial q^\nu}$$

F's work:

$$\mathrm{d}W = \sum_i F_i\,\mathrm{d}x^i = \sum_\nu Q_\nu\,\mathrm{d}q^\nu + K\,\mathrm{d}t \quad \text{where } Q_\nu = \sum_i F_i\,\frac{\partial x_i}{\partial q^\nu}.$$

Q_ν is called the generalised force.

$$T = \frac{1}{2}\sum_i m_i\,\dot{x}_i^2 = \frac{1}{2}\left[\sum_{\mu,\nu} T_{\mu\nu}\dot{q}^\mu\dot{q}^\nu + \sum_\mu A_\mu\dot{q}^\mu + B\right] = T(q, \dot{q}, t)$$

$$T_{\mu\nu} = \sum_i m_i\,\frac{\partial x_i}{\partial q^\mu}\,\frac{\partial x_i}{\partial q^\nu} = T_{\mu\nu}(q, t)$$

$$A_\mu = \sum_i \frac{\partial x_i}{\partial q^\mu}\,\frac{\partial x_i}{\partial t}$$

$$B = \sum_i m_i\left(\frac{\partial x_i}{\partial t}\right)^2$$

Generalised momentum p_μ is defined as

$$p_\mu = \frac{\partial T}{\partial \dot{q}^\mu} = \sum_i \frac{\partial T}{\partial \dot{x}_i}\,\frac{\partial \dot{x}_i}{\partial \dot{q}^n} = \sum_i \frac{\partial T}{\partial \dot{x}^i}\,\frac{\partial x_i}{\partial q^\mu}$$

Newton 2: $m_i \ddot{x_i} = F_i$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{x}_i} = F_i$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_\nu = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\nu} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial T}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}^\nu}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{\partial T}{\partial \dot{x}_i}\frac{\partial x_i}{\partial q^\nu}\right) =$$

$$= \sum_i \left(\dot{p}_i \frac{\partial x_i}{\partial q^\nu} + \frac{\partial T}{\partial \dot{x}_i}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x_i}{\partial q^\nu}\right) = \sum_i F_i \frac{\partial x_i}{\partial q^\nu} + \frac{\partial T(q, \dot{q}, t)}{\partial q^\nu} = Q_\nu + \frac{\partial T}{\partial q^\nu}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\nu} - \frac{\partial T}{\partial q^\nu} = Q_\nu}$$

This is Lagrange's equations without a potential.

If the force is conservative:

$$F_i = -\frac{\partial V(x)}{\partial x_i}$$

$$Q_\nu = -\sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q^\nu} = -\frac{\partial V(q,t)}{\partial q^\nu}, \quad V(q,t) \equiv V(x(q,t))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\nu} - \frac{\partial T}{\partial q^\nu} = -\frac{\partial V}{\partial q^\nu}$$

or

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\nu} - \frac{\partial L}{\partial q^\nu} = 0}$$

where $L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t)$ called the Lagrangian function of the mechanical system.

Example: planetary motion. Use polar coordinates (r, θ).

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$V = -\frac{m K}{r}$$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{m K}{r}$$

Equations:

$$r\colon \quad \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - m r \dot{\theta}^2 + \frac{m K}{r^2} = 0$$

$$\theta\colon \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(m r^2 \dot{\theta}\right) = 0 \quad \Rightarrow \quad m r^2 \dot{\theta} = \text{constant (angular momentum is conserved)}$$

Example 2: plane pendulum (1 degree of freedom).

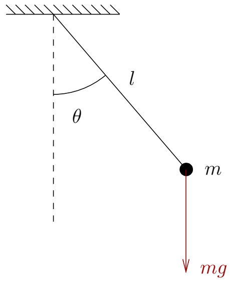

Figure 1.

$$L = T - V_{\text{grav}} = \frac{1}{2} m \left(l \dot{\theta}\right)^2 - m g h = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos\theta$$

Equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m l^2 \dot{\theta}\right) + m g l \sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0, \quad \text{the familiar pendulum equation}$$

The same recipe works for the double pendulum, although there is no description with n cartesian coordinates. Why? Because the recipe works in the presence of *holonomic constraints*.

DEFINITION: A holonomic constraint is a constraint in the possible motions of the system in the form $f(x_1, x_2, \ldots, x_n, t) = 0$.

EXAMPLE: Plane pendulum in Cartesian coordinates x, y. We have one holonomic constraint: $x^2 + y^2 - l^2 = 0$.

EXAMPLE: A rigid body consists of many atoms and many constraints $|\boldsymbol{r}_i - \boldsymbol{r}_j| - d_{ij} = 0$.

In general a system of N particles in three dimensions, restricted by m holonomic constraints, moves $3N - m$ manifold, has $3N - m \equiv n$ degrees of freedom, needs n general coordinates. For the pendulum $n = 2 \cdot 1 - 1 = 1$.

How to proceed? Constraints require constraint forces, constraining motion, otherwise as before.

$$m_i \ddot{x}_i = F_i = F_i^{\text{appl}} + F_i^{\text{constr}}$$

In the example of the pendulum the tension $\boldsymbol{S}$ in the coord is the constraint force, and mg is the applied force. Note that $\boldsymbol{S}$ does no work, $\boldsymbol{S} \cdot \delta \boldsymbol{r} = 0$, for all possible infinitesimal motions of the pendulum.

General principle:

$$\mathrm{d}W^{\text{constr}} = \sum_i F_i^{\text{constr}} \mathrm{d}x_i = \sum_\nu Q_\nu^{\text{constr}} \mathrm{d}q^\nu = 0$$

for all instantaneous motions $\mathrm{d}q^\nu$ of the system compatible with the constraints, i.e., such that d $f_a(q,t) \equiv \sum_\nu \frac{\partial f_\alpha}{\partial q^\nu} \mathrm{d}q^\nu = 0$. The constraints were $f_a(q,t) = 0$ for $\alpha = 1, \ldots, m$.

Principle:

$$\sum_i \left(m \ddot{x}_i - F_i^{\text{appl}} - F_i^{\text{constr}}\right) \mathrm{d}x_i = 0$$

$$\sum_\nu \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}^\nu} - \frac{\partial T}{\partial \dot{q}^\nu} - Q_\nu^{\text{appl}} - Q_\nu^{\text{constr}}\right) \mathrm{d}q^\nu = 0$$

for all instantaneous $\mathrm{d}q^\nu$'s compatible with the constraints, i.e.

$$\sum_i \frac{\partial f_\alpha}{\partial x_i}\mathrm{d}x_i = 0 \quad \text{or} \quad \sum_\nu \frac{\partial f_\alpha}{\partial q^\nu}\mathrm{d}q^\nu = 0$$

Principle:

$$\sum_{i=1}^{3N} \left(F_i^{\text{appl}} - m_i \ddot{x}_i\right)\mathrm{d}x_i = 0$$

for all $\mathrm{d}x_1$ such that $\sum_{i=1}^{3N} \frac{\partial f_\alpha}{\partial x_i}\mathrm{d}x_i = 0$. This is d'Alembert's principle. An axiom of mechanics.

$$\Leftrightarrow$$

$$\sum_{\nu=1}^{3N} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\nu} - \frac{\partial T}{\partial q^\nu} - Q_\nu^{\text{appl}}\right)\mathrm{d}q^\nu = 0$$

for all $\mathrm{d}q^\nu$ such that

$$\sum_{\nu=1}^{3N} \frac{\partial f_\alpha(q,t)}{\partial q^\nu}\mathrm{d}q^\nu = 0.$$

Now one can choose the constraints $f_\alpha(q,t) = 0$, $\alpha = 1, \ldots, m$ to solve for m general coordinates. There remain $n = 3N - m$ independent equations. This gives n Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\nu} - \frac{\partial T}{\partial q^\nu} = Q_\nu^{\text{appl}}$$