2008–09–02

L1. Repetition of Newton's Form of Mechanics.

⟨fig1⟩

The particle's position is described by position vector $\boldsymbol{r}(t) = (x(t), y(t))$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t) = \dot{\boldsymbol{r}}(t) = \text{velocity vector}$$

$$\boldsymbol{p} = m\,\boldsymbol{v}(t) = \text{particle's linear momomentum}$$

Newton 2:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} \equiv \dot{\boldsymbol{p}} = \boldsymbol{F}$$

Almost always $\dot{\boldsymbol{p}} = m\,\ddot{\boldsymbol{r}}$.

In Cartesian coordinates there is one equation of motion for each degree of freedom.

$$\begin{cases} m\,\ddot{x} = F_x \\ m\,\ddot{y} = F_y \\ \vdots \end{cases}$$

Particle systems

Newton 2:

$$m\,\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i, \quad i = 1, \ldots, N, \quad \text{where } N \text{ is the number of particles.}$$

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{ex}} + \sum_{j \neq i} \boldsymbol{F}_{ji}$$

Newton 3: $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = 0$: law of action and reaction (weak form). The strong form has the additional requirement $(\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ji} = \boldsymbol{0}$. ⟨fig2⟩

Conservation laws:

Momentum: $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$

$$\dot{\boldsymbol{P}} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \left(\boldsymbol{F}_i^{\text{ex}} + \sum_{j \neq i} \boldsymbol{F}_{ji}\right) = \sum_i \boldsymbol{F}_i^{\text{ex}} \equiv \boldsymbol{F}^{\text{ex}}$$

$\dot{\boldsymbol{P}} = \boldsymbol{F}^{\text{ex}}$ — fundamental in rigid body mechanics.

Special case: for isolated systems $\boldsymbol{P}(t) = \text{constant}$.

Angular momentum: $\boldsymbol{L} = \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i\,\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i$.

$$\dot{\boldsymbol{L}} = \sum_i m_i\Big(\underbrace{\dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i}_{=0} + \dot{\boldsymbol{r}}_i \times \ddot{\boldsymbol{r}}_i\Big) = \sum_i \left(\boldsymbol{r}_i \times \left(\boldsymbol{F}_i^{\text{ex}} + \sum_{j \neq i} \boldsymbol{F}_{ji}\right)\right)$$

$$\sum_{\substack{i,j \\ i \neq j}} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = [\text{Newton 3}] = -\sum_{i,j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = -\sum_{i,j} \boldsymbol{r}_j \times \boldsymbol{F}_{ji} = \frac{1}{2}\sum_{i,j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ji} = 0$$

according to the strong form of Newton's third law.

$$\dot{\boldsymbol{L}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{\mathrm{ex}} \equiv \sum_i \boldsymbol{\tau}_i^{\mathrm{ex}} = \boldsymbol{T}^{\mathrm{ex}}$$

For an isolated system the total angular momentum is constant in time.

Energy for one particle:

Newton 2: $m\,\ddot{\boldsymbol{r}} = \boldsymbol{F}$.

$$T \equiv \frac{1}{2} m\,\dot{\boldsymbol{r}}^2 \quad \Rightarrow \quad \dot{T} = m\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \boldsymbol{F} \cdot \dot{\boldsymbol{r}}$$

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = W_{12}$$

If $\boldsymbol{F}$ is conservative; i.e., $\boldsymbol{F} = -\nabla V(\boldsymbol{r})$, then

$$\int_{t_1}^{t_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -\int_{t_1}^{t_2} \nabla V(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = -\left(V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1)\right)$$

$$\Rightarrow T_2 + V_2 = T_1 + V_1 = E = \text{constant in time.}$$

There are three equivalent criteria for conservative forces:

1. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$
2. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$ and $\oint_i \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$.
3. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$ and $\nabla \times \boldsymbol{F} = 0$. (And the region has to be simply connected.)

Potential for particle systems

$$T_2 - T_1 = \int_{t_1}^{t_2} \sum_i \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i = \int_{t_1}^{t_2} \sum_i \mathrm{d}\boldsymbol{r}_i \cdot \left(\boldsymbol{F}_i^{\mathrm{ex}} + \sum_{j \neq i} \boldsymbol{F}_{ji}\right)$$

Assume $\boldsymbol{F}_i^{\mathrm{ex}} = -\nabla_i V_i^{\mathrm{ex}}(\boldsymbol{r}_i)$ and $\boldsymbol{F}_{ji} = -\nabla_i V_i(\boldsymbol{r}_i - \boldsymbol{r}_j)$

Newton 3 (weak): $0 = F_{ji} + F_{ij} = -\nabla_i(V_i(\boldsymbol{r}_i - \boldsymbol{r}_j) - V_j(\boldsymbol{r}_j - \boldsymbol{r}_i))$

$$V_i(\boldsymbol{r}_i - \boldsymbol{r}_j) = V_j(\boldsymbol{r}_j - \boldsymbol{r}_i)$$

Newton 3 (strong): $V_i = V_i(|\boldsymbol{r}_i - \boldsymbol{r}_j|) = V_j(|\boldsymbol{r}_j - \boldsymbol{r}_i|)$.

Example: if all particles are similar and all V_i's are the same

$$\boldsymbol{F}_i^{\mathrm{ex}} = -\nabla V^{\mathrm{ex}}(\boldsymbol{r}_i)$$

$$\boldsymbol{F}_{ji} = -\nabla_i V^{\mathrm{int}}(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$$

then

$$E = \sum_i T_i + \sum_i V^{\mathrm{ex}}(\boldsymbol{r}_i) + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V^{\mathrm{int}}(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$$

Example: gas of electrons.

Towards rigid body mechanics.

For a particle system, define the centre of mass coordinates $\boldsymbol{R}$ and the coordinates relative to the centre of mass.

$$\boldsymbol{R} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_k m_k} = \frac{1}{M} \sum_i m_i \, \boldsymbol{r}_i$$

$$\Rightarrow M \boldsymbol{R} = \sum_i m_i \, \boldsymbol{r}_i$$

$$M\dot{\boldsymbol{R}} = \sum_i m_i \dot{\boldsymbol{r}}_i = \sum_i \boldsymbol{p}_i = \boldsymbol{P}$$

$$M\ddot{\boldsymbol{R}} = \boldsymbol{F}^{\text{ex}}$$

$$\boldsymbol{r}_i = \boldsymbol{R} + \boldsymbol{r}_i' \quad \text{(this defines } \boldsymbol{r}_i'\text{)}$$

⟨fig3⟩.

$$\sum_i m_i \, \boldsymbol{r}_i' = 0$$

$$\boldsymbol{L} = \sum_i m_i \big(\boldsymbol{R} + \boldsymbol{r}_i'\big) \times \Big(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}_i'\Big) =$$

$$\sum_i \left(m_i \boldsymbol{R} \times \dot{\boldsymbol{R}} + \underbrace{m_i \boldsymbol{R} \times \dot{\boldsymbol{r}}_i'}_{=0} + \underbrace{m_i \boldsymbol{r}_i' \times \boldsymbol{R}}_{=0} + m_i \, \boldsymbol{r}_i' \times \dot{\boldsymbol{r}}_i' \right) = M \, \boldsymbol{R} \times \dot{\boldsymbol{R}} + \sum_i m_i \boldsymbol{r}_i' \times \dot{\boldsymbol{r}}_i'$$

The angular momentum of the system equals the angular momentum of a particle at $\boldsymbol{R}$ of mass M, plus the angular momentum due to motion relative to the centre of mass.

$$T = \frac{1}{2} \sum_i m_i \dot{\boldsymbol{r}_i}^2 = \frac{1}{2} \sum_i m_i \Big(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}}_i'\Big)^2 = \frac{1}{2} \sum_i \left(m_i \dot{\boldsymbol{R}}^2 + \underbrace{m_i \cdot 2\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{r}}_i'}_{=0} + m_i (\dot{\boldsymbol{r}}_i')^2 \right) =$$

$$= \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2} \sum_i m_i (\dot{\boldsymbol{r}}_i)^2$$

Kinetic energy for system = kinetic energy for a particle of mass M at $\boldsymbol{R}$ + kinetic energy due to motion relative to the centre of mass.

T and L for a rigid body:

For a rigid body $\dot{\boldsymbol{r}}_i' = \boldsymbol{\omega} \times \boldsymbol{r}_i'$, where $\boldsymbol{\omega}$ is the rotation vector of the rigid body, a vector pointing along the rotation axis, of length = angular velocity.

⟨fig4⟩

$$T = \frac{1}{2} M \dot{R}^2 + \sum_i \frac{1}{2} m_i (\boldsymbol{\omega} \times \boldsymbol{r}')^2 = \frac{1}{2} M \dot{R}^2 + \sum_i \frac{1}{2} m_i \Big(\omega^2 (r_i')^2 - (\boldsymbol{\omega} \cdot \boldsymbol{r}_i')^2 \Big) =$$

$$= \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \sum_{\substack{a \in \{1,2,3\} \\ b \in \{1,2,3\}}} \omega_a \, I_{ab} \, \omega_b$$

This is the definition of the inertia tensor I_{ab}.

$$I_{ab} = \sum_i m_i\big((r_i')^2\,\delta_{ab} - r_{ia}' r_{ib}'\big) = \text{inertia tensor with respect to the centre of mass.}$$

Special case: If the axis of rotation is fixed, $\hat{\boldsymbol{\omega}}$ is fixed ($\boldsymbol{\omega} = \omega\,\hat{\boldsymbol{\omega}}$), then

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} I \omega^2$$

where $I = \hat{\omega}_a\, I_{ab}\, \hat{\omega}_b$.

Similarly for $\boldsymbol{L}$:

$$\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P} + \sum_i m_i(\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i) = \sum_i m_i\big(\boldsymbol{r}_i' \times (\boldsymbol{\omega} \times \boldsymbol{r}_i')\big) =$$

$$= \sum_i m_i\Big(\big(\boldsymbol{r}_i'\big)^2 \boldsymbol{\omega} - \boldsymbol{r}_i'\,(\boldsymbol{r}_i' \cdot \boldsymbol{\omega})\Big)$$

$$L_a = (\boldsymbol{R} \times \boldsymbol{P})_a + \sum_{b=1}^{3} I_{ab}\,\omega_b$$